UNIVERSITY OF LONDON | INTERNATIONAL PROGRAMMES

Course information 2015–16
MT105a Mathematics 1 (half course)

This half course develops basic mathematical methods and will emphasise their applications to problems in economics, management and related areas.

Prerequisite
None apply.

Exclusion
This half course may not be taken with:
MT1173 Algebra
MT1174 Calculus

Aims and objectives
The objectives specifically include:
- To enable students to acquire skills in the methods of calculus (including multivariate calculus) and linear algebra, as required for their use in economics-based subjects.
- To prepare students for further units in mathematics and/or related disciplines.

Essential reading
For full details please refer to the reading list.
Anthony, M. and N. Biggs *Mathematics for Economics and Finance*. (Cambridge: Cambridge University Press)

Assessment
This half course is assessed by a two hour unseen written examination.

Learning outcomes
At the end of this half course and having completed the essential reading and activities students should have:

- ☑ used the concepts, terminology, methods and conventions covered in the half course to solve mathematical problems in this subject.
- ☑ the ability to solve unseen mathematical problems involving understanding of these concepts and application of these methods
- ☑ seen how mathematical techniques can be used to solve problems in economics and related subjects

Students should consult the *Programme Regulations for degrees and diplomas in Economics, Management, Finance and the Social Sciences* that are reviewed annually. Notice is also given in the *Regulations* of any courses which are being phased out and students are advised to check course availability.

Syllabus

This is a description of the material to be examined, as published in the *Programme handbook*. On registration, students will receive a detailed subject guide which provides a framework for covering the topics in the syllabus and directions to the essential reading.

This half course develops basic mathematical methods and will emphasise their applications to problems in economics, management and related areas.

Basics: Basic algebra; Sets, functions and graphs; Factorisation (including cubics); Inverse and composite functions; Exponential and logarithm functions; Trigonometrical functions.

Differentiation: The meaning of the derivative; Standard derivatives; Product rule, quotient rule and chain rule; Optimisation; Curve sketching; Economic applications of the derivative: marginals and profit maximisation.

Integration: Indefinite integrals; Definite integrals; Standard integrals; Substitution method; Integration by parts; Partial fractions; Economic applications of integration: determination of total cost from marginal cost, and cumulative changes.

Functions of several variables: Partial differentiation; Implicit partial differentiation; Critical points and their natures; Optimisation; Economic applications of optimisation; Constrained optimisation and the Lagrange multiplier method; The meaning of the Lagrange multiplier; Economic applications of constrained optimisation.

Matrices and linear equations: Vectors and matrices, and their algebra; Systems of linear equations and their expression in matrix form; Solving systems of linear equations using row operations (in the case where there is a unique solution); Some economic/managerial applications of linear equations.

Sequences and series: Arithmetic and Geometric Progressions; Some Financial application of sequences and series.

Examiners' commentaries 2015

MT105a Mathematics 1

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2014–15. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Please note carefully the comment in relation to Question 3 concerning possible future requirement to use row operations to solve systems of linear equations.

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions – Zone A

Candidates should answer all **EIGHT** questions: all **SIX** questions of Section A (60 marks in total) and **BOTH** questions from Section B (20 marks each).

Section A

Answer all **six** questions from this section (60 marks in total).

Question 1

A monopolist's marginal cost function is given by

$$MC = 20 + 4q,$$

where q is the quantity of good produced. Her fixed costs are 20, and the demand equation for the good she produces is

$$p + 4q = 40,$$

where p and q are price and quantity, respectively. Find expressions for the total revenue and for the profit, as functions of q. Determine the value of q which maximises the profit.

Reading for this question

See Chapter 3 of the subject guide for related reading.

Approaching the question

The total cost is $TC = \int MC \, dq$, which is $2q^2 + 20q + c$. Since $FC = 20 = TC(0)$, we have $c = 20$. You should explain, as here, why the constant is 20 and not simply assume that it must be. The monopolist's demand equation is $p + 4q = 40$, so when the quantity produced is q, the selling price will be $p = 40 - 4q$ and hence the total revenue will be $TR = qp = q(40 - 4q) = 40q - 4q^2$. So the profit function is:

$$\Pi(q) = TR(q) - TC(q) = 40q - 4q^2 - (2q^2 + 20q + 20) = -6q^2 + 20q - 20.$$

We solve $\Pi'(q) = 0$. We have $\Pi'(q) = 20 - 12q$, and this is 0 when $q = 5/3$. This gives a maximum because $\Pi''(q) = -12 < 0$ (or because Π' changes sign from positive to negative). (We could simply note that Π is a quadratic with negative squared term and hence the critical point must be a maximum.)

Question 2

Use the Lagrange multiplier method to determine the values of x and y that minimise $x + 2y$ subject to the constraint $x^3y^2 = 27$.

Reading for this question

The Lagrange multiplier method for constrained optimisation is discussed in Chapter 5 of the subject guide.

Approaching the question

The Lagrangian is $L = x + 2y - \lambda(x^3y^2 - 27)$. The first order conditions are:

$$1 - 3\lambda x^2y^2 = 0,$$

$$2 - 2\lambda x^3y = 0,$$

$$x^3y^2 = 27.$$

The first two equations imply that $\dfrac{1}{3x^2y^2} = \dfrac{1}{x^3y}$, and so $x = 3y$. (It's clear that the constraint is not satisfied when x or y is 0, so we can assume they are non-zero.)

Then, $(3y)^3y^2 = 27$, which is $y^5 = 1$, so:

$$x = 3, y = 1.$$

Question 3

Suppose the numbers x, y, z satisfy the following equations, where a is some number:

$$\begin{aligned} x + y + z &= 5 - a \\ 2x + y &= 7 - a \\ x - y - 2z &= 0. \end{aligned}$$

Use a matrix method to determine the values of x, y and z, in terms of a.

For what values of a will x, y, z all be positive?

Reading for this question

The recommended method for solving linear equations can be found in Chapter 6 of the subject guide.

Approaching the question

It is known by several names: the row operation method, the Gauss–Jordan method, the row-reduction method, and so on. Many candidates like to use a different method, not covered in the subject guide, especially 'Cramer's rule'. It is acceptable to do so in this question, since it only requires the use of a matrix method. (But, it has to be a *matrix* method, as the question makes explicit: manipulation of the equations will not suffice.) However, our view is that the row operations method is easier and less prone to error, and it is the method that we intend students to learn. **It is possible that in future years we might insist explicitly that a row operations method be used, in which case using Cramer's rule or some other method would not be answering the question.** If we require the use of the row operation method we shall use the phrase 'Use row operations'.

The augmented matrix is:

$$(A\,|\,\mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 & 5-a \\ 2 & 1 & 0 & 7-a \\ 1 & -1 & -2 & 0 \end{pmatrix}$$

A valid reduction to row-echelon form is as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 5-a \\ 2 & 1 & 0 & 7-a \\ 1 & -1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 5-a \\ 0 & -1 & -2 & -3+a \\ 0 & -2 & -3 & a-5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 5-a \\ 0 & 1 & 2 & 3-a \\ 0 & 0 & 1 & 1-a \end{pmatrix}.$$

(You can stop here, or continue reducing further, as some do, until the first three columns form the identity matrix.)

So, from the row-echelon form, we have:

$$\begin{aligned} z &= 1-a, \\ y &= 3-a-2(1-a) = 1+a, \\ x &= 5-a-(1+a)-(1-a)-1 = 3-a. \end{aligned}$$

For x, y, z all to be positive, we must have $1-a, 1+a, 3-a > 0$ which means $a < 1$, $a > -1$ and $a < 3$. So we need $-1 < a < 1$.

This is probably a good point at which to make some general comments about how questions are marked. Clearly, in a question like this, it is easy to get the wrong answer. (Though it should be noted that in this particular question, you can always substitute the values that you have found into the original equations, and this will show whether these are correct or not. So you can tell if you have the wrong answer and, if you have time, you can then re-work the calculation.) Examiners understand that arithmetical errors can be made, especially in the stressful circumstances of an examination. Quite probably, the Examiners themselves would make some mistakes if they sat the paper. So, although there are marks for correct calculation, there are also marks for using the right method (even if you make a mistake). So, here, for instance, the Examiners will award marks if you can indicate that you know how to start to solve the equations (by writing down an augmented matrix); that you know what row operations are; that you know what it is you want to achieve with row operations (the reduced matrix, that is); and that you then know how to work from that reduced matrix to determine the required solutions. There are marks for all these things.

Be sure to understand that only certain types of operations qualify as valid row operations. In particular, a number of candidates make the mistake of thinking that subtracting a fixed constant from each entry of a row is valid. It is not. (And, if you don't know what we mean by that, then you're probably not doing it, which is good!)

Question 4

The function f is defined by

$$\boldsymbol{f(x,y) = \frac{1}{\sqrt{x^2+y^2}}.}$$

Show that

$$\boldsymbol{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^{1/3} = f(x,y).}$$

Reading for this question

Partial differentiation is discussed in Chapter 5 of the subject guide.

Approaching the question

It is important in answering questions like this to use a correct notation for partial derivatives. We can write f_x instead of $\frac{\partial f}{\partial x}$ and f_{xx} instead of $\frac{\partial^2 f}{\partial x^2}$, and so on, but it is a bad idea to invent your own notation! In particular, writing $f(x,x)$ or $f'(x,x)$ for $\frac{\partial^2 f}{\partial x^2}$ is just plain wrong.

We have:

$$f_x = \frac{-x}{(x^2+y^2)^{3/2}},$$

$$f_y = \frac{-y}{(x^2+y^2)^{3/2}},$$

$$f_{xx} = \frac{-(x^2+y^2)^{3/2} + 3x^2(x^2+y^2)^{1/2}}{(x^2+y^2)^3} = \frac{-(x^2+y^2)+3x^2}{(x^2+y^2)^{5/2}} = \frac{2x^2-y^2}{(x^2+y^2)^{5/2}},$$

$$f_{yy} = \frac{2y^2-x^2}{(x^2+y^2)^{5/2}}.$$

Then:

$$\begin{aligned} f_{xx}+f_{yy} &= \frac{2x^2-y^2+2y^2-x^2}{(x^2+y^2)^{5/2}} = \frac{x^2+y^2}{(x^2+y^2)^{5/2}} = \frac{1}{(x^2+y^2)^{3/2}} \\ &= (f(x,y))^3. \end{aligned}$$

So:

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)^{1/3} = f(x,y).$$

Question 5

Show that the function f given by

$$\boldsymbol{f(x,y) = x^4 + 2x^2y + 2y^2 + y}$$

has three critical points. For each critical point of f, determine whether it is a local minimum, local maximum, or saddle point.

Reading for this question

This question uses the material in Chapter 5 of the subject guide.

Approaching the question

In order to find the critical points of the function:

$$f(x, y) = x^4 + 2x^2y + 2y^2 + y$$

we work out the partial derivatives of f:

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy, \quad \frac{\partial f}{\partial y} = 2x^2 + 4y + 1.$$

The critical points are given by the first-order conditions:

$$4x^3 + 4xy = 0, \quad 2x^2 + 4y + 1 = 0.$$

The first of these equations says that $x(x^2 + y) = 0$, and so either (i) $x = 0$ or (ii) $y = -x^2$.

We now analyse each of the two cases separately.

(i) When $x = 0$, the second equation is $4y + 1 = 0$, giving $y = -1/4$. Thus one critical point is $(0, -1/4)$.

(ii) When $y = -x^2$ the second equation becomes $2x^2 - 4x^2 + 1 = 0$, and so x is either $1/\sqrt{2}$ or $-1/\sqrt{2}$. The corresponding values of y are determined by $y = -x^2$, and are both $-1/2$.

Thus, there are three critical points: $(0, -1/4)$, $(1/\sqrt{2}, -1/2)$ and $(-1/\sqrt{2}, -1/2)$.

To classify them we need the second partial derivatives of f:

$$f_{xx}(x, y) = 12x^2 + 4y, \quad f_{xy}(x, y) = 4x, \quad f_{yy}(x, y) = 4.$$

Note that, as is generally the case, these are not constant, so we have to look at each critical point individually.

At $(0, -1/4)$: we have $f_{xx} = -1$, $f_{xy} = 0$, $f_{yy} = 4$, so $D = (-1)(4) - 0 < 0$. Therefore $(0, -1/4)$ is a saddle point.

At $(1/\sqrt{2}, -1/2)$: we have $f_{xx} = 4$, $f_{xy} = 2\sqrt{2}$, $f_{yy} = 4$, so $f_{xx} > 0$ and $D = (4)(4) - 8 > 0$. Therefore $(1/\sqrt{2}, -1/2)$ is a local minimum.

At $(-1/\sqrt{2}, -1/2)$: we have $f_{xx} = 4$, $f_{xy} = -2\sqrt{2}$, $f_{yy} = 4$, so $f_{xx} > 0$ and $D = (4)(4) - 8 > 0$. Therefore $(-1/\sqrt{2}, -1/2)$ is also a local minimum.

There are other ways of using the equations to derive the critical points. Here's an alternative:

We have $4y = -1 - 2x^2$ from the second equation. Substituting into the first equation then gives $4x^3 - x - 2x^3 = 0$ so that $2x^3 - x = 0$. So $x(2x^2 - 1) = 0$ and therefore $x = 0$ or $x = \pm 1/\sqrt{2}$. When $x = 0$, $y = -1/4$ and when $x = \pm 1/\sqrt{2}$, $y = -1/2$.

Question 6

Sarah saves money in a bank account paying interest at a fixed rate of 5%, where the interest is paid once per year, at the end of the year. She deposits an amount A at the beginning of each of the next N years. Find an expression, in terms of A and N, and in as simple a form as possible, for the final amount saved (the amount just after the last deposit).

Reading for this question

Chapter 7 of the subject guide gives the required background material.

Approaching the question

There are two steps in questions like this. The first is to model the situation that is being described using mathematics and the second is to use that mathematics to find the answer. In this case, we are asked to find out something about the final amount saved, i.e. the amount in the account just after the last deposit, and so it makes sense to model this situation by seeing what the balance is **after each deposit**. This means that we may want to let y_n (say) represent the amount of money after the nth deposit and then, using the information in the question, we can figure out what the first few terms of this sequence (say y_1, y_2 and y_3) are. Having done this, we should be able to spot a pattern and, by generalising what we are seeing, we should be able to find what we want, i.e. y_N, the balance after her Nth (or last) deposit.

All this means is that a good solution to this question would run as follows if we bear in mind how compound interest works and what the question tells us about Sarah's savings.

Let y_n be the amount saved after the nth deposit.

This means that we have:

$$\begin{aligned}
&\text{(Amount after 1st deposit)} \quad y_1 = A \\
&\text{(Amount after 2nd deposit)} \quad y_2 = 1.05y_1 + A = 1.05A + A \\
&\text{(Amount after 3rd deposit)} \quad y_3 = 1.05y_2 + A = 1.05^2A + 1.05A + A
\end{aligned}$$

(Note that it is absolutely crucial that you specify what each of the items in this list represents either in words (as we have done in the brackets) or by specifying exactly what y_n is supposed to represent. If you don't do this, then no-one knows what you are trying to work out!)

We now look carefully at this list to try and spot a pattern and, if we can, we should see that it generalises to give us:

$$\text{(Amount after } N\text{th deposit)} \quad y_N = 1.05^N A + 1.05^{N-1}A + \cdots + 1.05A + A,$$

and this is what we were after, i.e. Sarah's balance after the Nth (or last) deposit. However, this is **not** the answer as it can be simplified. In particular, this is a geometric series with first term A, commmon ratio 1.05 and N terms so it can be summed, using the formula in the subject guide, to get:

$$\text{(Amount after } N\text{th deposit)} \quad y_N = A\frac{1.05^N - 1}{1.05 - 1}.$$

(Note that we can only apply the formula if our series for y_N is clearly specified. That is, we need to know how it starts, how it changes and, to get the number of terms, how it finishes!)

So, tidying this up, we can see that our final answer is:

$$\text{(Amount after } N\text{th deposit)} \quad y_N = 20A(1.05^N - 1).$$

As always, it's important to simplify the answer as much as possible!

Section B

Answer **both** questions from this section (20 marks each).

Question 7

(a) **The market value of a painting, if it is sold at time $t \geq 0$, is assumed to be $V(t) = (t+2)^2$. The *present value* $P(t)$ of money raised if it is sold at time t is $V(t)e^{-rt}$ where r is a constant such that $0 < r < 1$. Determine the value of t that maximises $P(t)$ and determine also the maximum value of $P(t)$.**

Approaching the question

We have:

$$P'(t) = 2(t+2)e^{-rt} + (t+2)^2 e^{-rt}(-r) = e^{-rt}(t+2)(2-2r-rt).$$

We solve $P'(t) = 0$ (for $t \geq 0$). For $t \geq 0$, $P'(t) = 0$ if and only if $t = T = (2-2r)/r$. (We reject $t = -2$ because it's negative.) Here, P' changes sign from positive to negative, so this maximises P. So the maximum value is the value at T, which is:

$$(T+2)^2 e^{-rT} = \left(\frac{2-2r}{r} + 2\right)^2 e^{-r(2-2r)/r} = \left(\frac{2}{r}\right)^2 e^{2r-2}.$$

It is also possible (though not as easy) to use the second-derivative test to establish that $t = T$ gives a maximum value. You should find that:

$$P''(t) = -re^{-rt}(t+2)(2-2r-rt) + e^{-rt}(2-2r-rt) - re^{-rt}(t+2).$$

So, $P''(T) = -re^{-rT}\left(\frac{2-2r}{r} + 2\right) = -2e^{-rT} < 0$, so $t = T$ gives a maximum.

(b) **Determine the integrals**

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x$$

$$\int_1^e x^2 \ln x\,\mathrm{d}x.$$

Reading for this question

Integration is discussed in Chapter 4 of the subject guide.

Approaching the question

It can be difficult because it is not always clear which technique will work. The three main techniques are: substitution, parts, and partial fractions. More than one method might work, and some integrals require a combination of methods.

We can approach the first integral by making the substitution $u = 1 + \sqrt{x}$. We then have $\mathrm{d}u = 1/(2\sqrt{x})\,\mathrm{d}x$, so that $\mathrm{d}x = 2\sqrt{x}\,\mathrm{d}u = 2(u-1)\,\mathrm{d}u$ and the integral becomes:

$$\int \frac{2(u-1)^2}{u}\,\mathrm{d}u.$$

This is:

$$2\int \left(u - 2 + \frac{1}{u}\right)\mathrm{d}u = u^2 - 4u + 2\ln u + c$$
$$= (1+\sqrt{x})^2 - 4(1+\sqrt{x}) + 2\ln(1+\sqrt{x}) + c.$$

The substitution $u = \sqrt{x}$ also works. With this choice, we have $\mathrm{d}u = 1/(2\sqrt{x})\,\mathrm{d}x$, so that $\mathrm{d}x = 2u\,\mathrm{d}u$ and the integral becomes:

$$\int \frac{2u^2}{1+u}\,\mathrm{d}u.$$

This is:

$$\int \frac{2u(1+u)}{1+u}\,\mathrm{d}u - \int \frac{2u}{1+u}\,\mathrm{d}u = \int 2u\,\mathrm{d}u - \int \frac{2(1+u)}{1+u}\,\mathrm{d}u + 2\int \frac{1}{1+u}\,\mathrm{d}u.$$

This is:

$$u^2 - 2u + 2\ln(1+u) + c = x - 2\sqrt{x} + 2\ln(1+\sqrt{x}) + c.$$

For the second integral, we have:

$$\begin{aligned}\int_1^e x^2 \ln x\,\mathrm{d}x &= \left[\frac{x^3}{3}\ln x\right]_1^e - \int_1^e \frac{x^2}{3}\,\mathrm{d}x \\ &= \frac{e^3}{3} - \left[\frac{x^3}{9}\right]_1^e \\ &= = \frac{e^3}{3} - \left(\frac{e^3}{9} - \frac{1}{9}\right) \\ &= \frac{2e^3}{9} + \frac{1}{9}.\end{aligned}$$

A correct answer must use the facts that $\ln 1 = 0$ and $\ln e = 1$.

Question 8

(a) **A firm has production function q given by $q(k,l) = k^{1/2}l^{1/4}$ where k denotes the amount of capital and l the amount of labour. Each unit of capital costs 10 dollars and each unit of labour costs 20 dollars. Use the Lagrange multiplier method to determine, in terms of q, the capital and labour that will minimise the cost of producing an amount q. Determine this minimum cost, simplifying your answer as much as possible.**

Approaching the question

We have to minimise $10k + 20l$ subject to the constraint $k^{1/2}l^{1/4} = q$. The Lagrangian is:

$$L = 10k + 20l - \lambda(k^{1/2}l^{1/4} - q),$$

and the optimal k and l satisfy the three equations:

$$\begin{aligned}10 - \frac{1}{2}\lambda k^{-1/2}l^{1/4} &= 0, \\ 20 - \frac{1}{4}\lambda k^{1/2}l^{-3/4} &= 0, \\ k^{1/2}l^{1/4} &= q.\end{aligned}$$

The first two equations, on elimination of λ, show that:

$$80k^{-1/2}l^{3/4} = 20k^{1/2}l^{-1/4},$$

so that $k = 4l$.

Then:

$$k^{1/2}l^{1/4} = (4l)^{1/2}l^{1/4} = q,$$

and:

$$l = \left(\frac{q}{2}\right)^{4/3},$$

so $k = 4l = 4\left(\frac{q}{2}\right)^{4/3}$. The minimum cost is:

$$10k + 20l = 40\left(\frac{q}{2}\right)^{4/3} + 20\left(\frac{q}{2}\right)^{4/3} = 60\left(\frac{q}{2}\right)^{4/3}.$$

(b) **Let n be any integer greater than 1. The function f is defined for positive x by**

$$\boldsymbol{f(x) = nx^{1/n} - x.}$$

Find the maximum value of $f(x)$ over all $x > 0$. Hence show that, for all $x > 0$,

$$\boldsymbol{x^{1/n} \leq \frac{x}{n} + \left(1 - \frac{1}{n}\right).}$$

Approaching the question

We have $f'(x) = x^{\frac{1}{n}-1} - 1$. We solve $f'(x) = 0$. The solution is $x = 1$. At $x = 1$, f' changes sign from positive to negative. This is because $\frac{1}{n} - 1 < 0$ and therefore if $x < 1$, $x^{\frac{1}{n}-1} > 1$ and if $x > 1$, $x^{\frac{1}{n}-1} < 1$. The point is therefore a maximum. Alternatively, $f''(x) = (\frac{1}{n} - 1)x^{\frac{1}{n}-2}$ so $f''(1) = (\frac{1}{n} - 1)$, and this is negative because $n > 1$. So the point is a maximum.

We therefore have that the maximum value among positive x is $f(1) = n - 1$. What that implies is that, for all $x > 0$:

$$f(x) \leq f(1) = n - 1$$

and hence:

$$nx^{1/n} - x \leq n - 1$$

and so:

$$x^{1/n} \leq \frac{x}{n} + \left(1 - \frac{1}{n}\right).$$