Course information 2015-16
MT105b Mathematics 2 (half course)

This half course develops further the basic mathematical methods introduced in MT1005a Mathematics 1, and also demonstrates further applications in economics, finance and management.

Rule
MT105a Mathematics 1 must be taken before, or at the same time as MT105b Mathematics 2.

Exclusion
This half course may not be taken with:
MT1173 Algebra
MT1174 Calculus
MT2076 Management Mathematics

Aims and objectives
This half course is designed to:
- enable students to acquire further skills in the methods of calculus and linear algebra (in addition to those in 05a Mathematics 1), as required for their use in economics-based subjects
- prepare students for further courses in mathematics and/or related disciplines.

Essential reading
For full details please refer to the reading list.

Anthony, M. and N. Biggs *Mathematics for Economics and Finance*. (Cambridge: Cambridge University Press)

Assessment
This half course is assessed by a two hour unseen written examination.

Learning outcomes
At the end of this half course and having completed the essential reading and activities students should have:

- ☑ used the concepts, terminology, methods and conventions covered in the half course to solve mathematical problems in this subject.
- ☑ the ability to solve unseen mathematical problems involving understanding of these concepts and application of these methods.
- ☑ seen how mathematical techniques can be used to solve problems in economics and related subjects

Students should consult the *Programme Regulations for degrees and diplomas in Economics, Management, Finance and the Social Sciences* that are reviewed annually. Notice is also given in the *Regulations* of any courses which are being phased out and students are advised to check course availability.

Syllabus

This is a description of the material to be examined, as published in the *Programme handbook*. On registration, students will receive a detailed subject guide which provides a framework for covering the topics in the syllabus and directions to the essential reading.

This course develops further the basic mathematical methods introduced in Mathematics 1, and also demonstrates further applications in economics, finance and management.

New techniques are also developed, particularly for linear algebra, differential equations and difference equations, and applications of these techniques are investigated.

Note: Mathematics 2 builds on Mathematics 1. Everything in the Mathematics 1 syllabus is needed for Mathematics 2. Thus, the Mathematics 2 syllabus includes the Mathematics 1 syllabus.

Further differentiation and integration: Mathematics 1 material on differentiation and integration; Using derivatives for approximations; Elasticities; Taylor's theorem; the effects of taxation; Definite integrals and the calculation of areas; Further economic applications of integration: includes consumer and producer surplus.

Functions of several variables: Mathematics 1 material on functions of several variables; Homogeneous functions and Euler's theorem; Review of constrained optimisation; Constrained optimisation for more than 2 variables; Further applications of constrained optimisation.

Linear Algebra: Mathematics 1 material on matrices and linear equations; Supply and demand, and the imposition of excise and percentage tax; Consistency of linear systems; Solving systems of linear equations using row operations, in the case where there are infinitely many solutions; Determinants and Cramer's rule; Calculation of inverse matrices by row operations; Economic applications of systems of linear equations, including input-output analysis; Eigenvalues and eigenvectors; Diagonalisation of matrices.

Differential equations: Exponential growth; Separable equations; Linear differential equations and integrating factors; Second-order differential equations; Coupled equations, including the use of matrix diagonalisation; Economic applications of differential equations.

Difference Equations: Solving first-order difference equations; Application of first-order difference equations to financial problems; The cobweb model; Second-order difference equations; Coupled first-order difference equations, including the use of matrix diagonalisation; Economic applications of second-order difference equations.

Examiners' commentaries 2015

MT105b Mathematics 2

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2014–15. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions — Zone A

Question 1

Given the inverse supply and demand functions

$$p^S(q) = 3q + 2 \qquad \text{and} \qquad p^D(q) = 6 - q,$$

respectively, we can easily find the equilibrium quantity, q^*, by solving the equation

$$p^S(q^*) = p^D(q^*) \quad \Longrightarrow \quad 3q^* + 2 = 6 - q^* \quad \Longrightarrow \quad 4q^* = 4 \quad \Longrightarrow \quad q^* = 1,$$

and the equilibrium price, p^*, can then be found by using, say, $p^* = p^D(q^*) = 6 - 1 = 5$.

If an excise tax of T is imposed on the market, then as in Section 2.6 of the subject guide, the supply function[1]

$$q^S(p) = \frac{p-2}{3} \quad \text{becomes} \quad q^S_T(p) = q^S(p - T) = \frac{p - T - 2}{3},$$

and the demand function[2] stays unchanged, i.e. we have

$$q^D_T(p) = q^D(p) = 6 - p.$$

[1]Of course, the inverse supply function, $p^S(q)$, means that the supply equation is $p = p^S(q)$ and so we have

$$p = 3q + 2 \quad \Longrightarrow \quad q = \frac{p-2}{3},$$

which is $q = q^S(p)$ in terms of the supply function, $q^S(p)$.

[2]Again, the inverse demand function, $p^D(q)$, means that the demand equation is $p = p^D(q)$ and so we have

$$p = 6 - q \quad \Longrightarrow \quad q = 6 - p,$$

which is $q = q^D(p)$ in terms of the demand function, $q^D(p)$.

The new equilibrium price, p_T^*, can then be found by solving the equation $q_T^S(p_T^*) = q_T^D(p_T^*)$, i.e.

$$\frac{p_T^* - T - 2}{3} = 6 - p_T^* \implies p_T^* - T - 2 = 18 - 3p_T^* \implies 4p_T^* = 20 + T,$$

which means that

$$p_T^* = 5 + \frac{T}{4}.$$

The new equilibrium quantity, q_T^*, can then be found using the demand function,[3] i.e. we have

$$q_T^* = q_T^D(p_T^*) = 6 - \left(5 + \frac{T}{4}\right) = 1 - \frac{T}{4}.$$

The tax revenue, R_T, is given by

$$R_T = Tq_T^* = T\left(1 - \frac{T}{4}\right) = T - \frac{T^2}{4},$$

and this has a stationary point when $R_T' = 0$, i.e. when

$$1 - \frac{T}{2} = 0 \implies T = 2,$$

and this is indeed the maximum we seek since $R_T'' = -1/2 < 0$. Thus, the tax revenue is maximised when $T = 2$.

When this excise tax is imposed, the total revenue generated by this market is $R = p_T^* q_T^*$ and, of that revenue, $R_T = Tq_T^*$ has to be paid in tax. This means that the suppliers only get to keep an amount given by

$$R - R_T = p_T^* q_T^* - Tq_T^* = (p_T^* - T)q_T^*.$$

Of course, in this case, we have $T = 3/2$ and so

$$p_T^* = 5 + \frac{2}{4} = 5 + \frac{1}{2} = \frac{11}{2} \quad \text{and} \quad q_T^* = 1 - \frac{2}{4} = 1 - \frac{1}{2} = \frac{1}{2},$$

which means that

$$R - R_T = \left(\frac{11}{2} - 2\right)\frac{1}{2} = \left(\frac{7}{2}\right)\frac{1}{2} = \frac{7}{4},$$

is the revenue that the suppliers get to keep.

Question 2

Given the positive constants k, a and b, the supply and demand functions for a particular good are given by

$$q^S(p) = p - b \quad \text{and} \quad q^D(p) = \frac{k}{p} - a,$$

respectively. So, as the equilibrium quantity is $a + b$, we can use the former to see that

$$a + b = p^* - b \implies p^* = a + 2b,$$

is the equilibrium price and then the latter to see that

$$a + b = \frac{k}{a + 2b} - a \implies \frac{k}{a + 2b} = 2a + b \implies k = (2a + b)(a + 2b),$$

gives us k in terms of a and b.

[3] A common error at this point is to use the original supply equation, i.e. $q = q^S(p)$, instead of the modified one, i.e. $q = q_T^S(p)$, to find the new equilibrium quantity. To avoid this mistake, it is usually a good idea to use the demand equation, as we do here, since this doesn't change!

As in Section 2.9 of the subject guide, the consumer surplus is given by

$$\text{CS} = \left(\int_0^{q^*} p^D(q)\,\mathrm{d}q \right) - p^*q^*.$$

So, since the demand equation is

$$q = \frac{k}{p} - a \implies \frac{k}{p} = q + a \implies p = \frac{k}{q+a} = p^D(q),$$

is the demand function and so we have

$$\int_0^{q^*} p^D(q)\,\mathrm{d}q = \int_0^{a+b} \frac{k}{q+a}\,\mathrm{d}q = \Big[k\ln(q+a) \Big]_0^{a+b} = k\Big(\ln(2a+b) - \ln(a) \Big) = k\ln\left(\frac{2a+b}{a}\right),$$

which means that, as $p^*q^* = (a+2b)(a+b)$, the consumer surplus is

$$\text{CS} = k\ln\left(\frac{2a+b}{a}\right) - (a+2b)(a+b) = (2a+b)(a+2b)\ln\left(\frac{2a+b}{a}\right) - (a+2b)(a+b),$$

in terms of a and b only.

Question 3

This question uses ideas from Section 2.4 of the subject guide to see that as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

we find that

$$\begin{aligned}\ln(1+\sin x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \cdots\right)^3 \\ - \frac{1}{4}(x - \cdots)^4 + \frac{1}{5}(x - \cdots)^5 + \cdots,\end{aligned}$$

if we keep the relevant terms from both series. Indeed, as we want to keep terms up to x^5, we can see that the brackets in the second term give us

$$\begin{aligned}\left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right) &= (x)(x) + 2(x)\left(-\frac{x^3}{3!}\right) + \cdots \\ &= x^2 - \frac{x^4}{3} + \cdots,\end{aligned}$$

and the brackets in the third term give us

$$\begin{aligned}\left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right) &= (x)(x)(x) + 3(x)(x)\left(-\frac{x^3}{3!}\right) + \cdots \\ &= x^3 - \frac{x^5}{2} + \cdots,\end{aligned}$$

whereas the brackets in the fourth and fifth terms give us $x^4 + \cdots$ and $x^5 + \cdots$ respectively. In particular, notice that here we're trying to make it clear that each term that arises from these products is obtained by multiplying out the relevant brackets so that we find all of the necessary terms. Overall then, we have

$$\begin{aligned}\ln(1+\sin x) = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right) + \frac{1}{3}\left(x^3 - \frac{x^5}{2} + \cdots\right) \\ - \frac{1}{4}\Big(x^4 + \cdots\Big) + \frac{1}{5}\Big(x^5 + \cdots\Big) + \cdots,\end{aligned}$$

and, gathering up the terms, this gives us

$$\ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + \cdots,$$

as the power series for $\ln(1+\sin x)$ in terms up to x^5.

Question 4

When using row operations to find the determinant of a matrix, as in Section 4.5.2 of the subject guide, if we only employ row operations of the form $R_i \to R_i + kR_j$ with $i \neq j$, the value of the determinant does not change. Doing this, we find that

$$A = \begin{pmatrix} 1 & -2 & 3 & 2 \\ 2 & -5 & 3 & 1 \\ 1 & 2 & 16 & 0 \\ 2 & -6 & 3 & 4 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 4 & 13 & -2 \\ 0 & -2 & -3 & 0 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 + 4R_2} \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & 1 & -14 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{R_4 \to R_4 - 3R_3} \begin{pmatrix} 1 & -4 & 3 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & 1 & -14 \\ 0 & 0 & 0 & 48 \end{pmatrix} = T.$$

As this is now an upper triangular matrix, the value of its determinant is just the product of its diagonal entries, i.e.

$$\det(T) = (1)(-1)(1)(48) = -48,$$

and, as $\det(A) = \det(T)$ because of the way we chose our row operations, we can see that $\det(A) = -48$ too.

Indeed, as this determinant is *non-zero*, this tells us that this matrix must be *invertible*.

Question 5

As in Section 5.2 of the subject guide, we see that the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Big(1 + f(t)\Big)\left(\frac{2t^2 - t + 1}{t}\right),$$

for $t \neq 0$ is separable and so, following the standard method, we rewrite it as

$$\int \frac{\mathrm{d}f}{1+f} = \int \frac{2t^2 - t + 1}{t}\,\mathrm{d}t \quad \Longrightarrow \quad \ln|1+f| = \int \left(2t - 1 + \frac{1}{t}\right) \mathrm{d}t = t^2 - t + \ln|t| + c,$$

where c is an arbitrary constant. Of course, this can be rewritten as

$$|1+f| = \mathrm{e}^{t^2 - t + \ln|t| + c} = \mathrm{e}^{t^2-t}\mathrm{e}^{\ln|t|}\mathrm{e}^c = A|t|\mathrm{e}^{t^2-t},$$

if we let $A = \mathrm{e}^c$ and so we find that

$$f(t) = At\mathrm{e}^{t^2-t} - 1,$$

is the general solution to this differential equation. But, we are also told that $f(1) = 2$ and so we can find the value of A by noting that this gives us $2 = A - 1$ so that $A = 3$. Consequently, we see that

$$f(t) = 3t\mathrm{e}^{t^2-t} - 1,$$

is the particular solution we seek.

Question 6

Following the method in Section 5.4 of the subject guide, we solve the second-order differential equation

$$f''(t) + 4f'(t) + 4f(t) = 8,$$

by first considering the auxiliary equation

$$m^2 + 4m + 4 = 0 \quad \Longrightarrow \quad (m+2)^2 = 0 \quad \Longrightarrow \quad m = -2, -2.$$

In this case, as we have one repeated real solution, the complementary function is

$$f_c(t) = (At + B)\mathrm{e}^{-2t},$$

for some arbitrary constants A and B. To find the particular integral, as the right-hand side is a constant, we try something of the form $f_p(t) = \alpha$ where α is a constant to be determined. Thus, $f'_p(t) = 0$ and $f''_p(t) = 0$ and, substituting these in to the given differential equation, we get

$$0 + 4(0) + 4(\alpha) = 8 \quad \Longrightarrow \quad \alpha = 2.$$

Consequently, the particular integral is $f_p(t) = 2$ and, adding this to the complementary function, we get

$$f(t) = f_c(t) + f_p(t) = (At + B)\mathrm{e}^{-2t} + 2.$$

as the general solution. Indeed, since $f(0) = 5$, we have

$$5 = B + 2 \quad \Longrightarrow \quad B = 3,$$

and since

$$f'(t) = A\mathrm{e}^{-2t} - 2(At + B)\mathrm{e}^{-2t},$$

we can use $f'(0) = -6$ as well to see that

$$-6 = A - 2(3) \quad \Longrightarrow \quad A = 0.$$

Thus, we find that

$$f(t) = 3\mathrm{e}^{-2t} + 2,$$

is the sought after particular solution. Indeed, we see that as t increases, $f(t)$ decreases to 2.[4]

Question 7

(a) There are two ways to find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{pmatrix}$.

- The first is the co-factor method given in Section 4.9 of the subject guide. To do this, we start by finding the determinant of the matrix which is

$$\det(A) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2(-2-1) - 1(-1-2) + 1(1-4) = -6,$$

if we expand along the first row (see Section 4.5 of the subject guide) and, as this is non-zero, this confirms that the matrix is invertible. We then find the matrix of minors, which is

$$\begin{pmatrix} -3 & -3 & -3 \\ -2 & -4 & 0 \\ -1 & 1 & 3 \end{pmatrix},$$

[4] Observe, in particular, that the *behaviour* of $f(t)$ as t increases is that it is *decreasing* to 2, its limit as $t \to \infty$.

where, for instance, the entry in the second row and first column is given by the determinant of the 2×2 matrix obtained from deleting the second row and first column of A, i.e.

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -2.$$

If we now change the signs in this matrix according to the sign convention, which we can write as

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix},$$

we then get the matrix of co-factors, i.e.

$$\begin{pmatrix} -3 & 3 & -3 \\ 2 & -4 & 0 \\ -1 & -1 & 3 \end{pmatrix}.$$

The inverse is then, simply, the transpose of this matrix divided by the determinant, i.e. we have

$$A^{-1} = -\frac{1}{6}\begin{pmatrix} -3 & 2 & -1 \\ 3 & -4 & -1 \\ -3 & 0 & 3 \end{pmatrix}.$$

- The second is the row operations method given in Section 4.10 of the subject guide. To do this we start with the augmented matrix

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right),$$

and then use row operations until this reduces to the form $(I\,|\,B)$. In this case, if we did everything correctly, we would find that B is the inverse of A, i.e. A^{-1}, as we found above.

(b) As in Section 3.3 of the subject guide, the function f given by

$$f(x, y, z) = \frac{x^{\alpha}y^{2\beta}z^{\gamma} - xy^2z^3}{x^{2\alpha}y^{\beta} + y^6z^{\gamma}} \ln\left(x^{2\alpha}y^{\beta}z^{\gamma}\right),$$

will be homogeneous as long as

- the term within the logarithm is homogeneous of degree zero, i.e. $2\alpha + \beta + \gamma = 0$.
- the numerator is homogeneous, i.e. $\alpha + 2\beta + \gamma = 1 + 2 + 3 \quad \Longrightarrow \quad \alpha + 2\beta + \gamma = 6$,
- the denominator is homogeneous, i.e. $2\alpha + \beta = 6 + \gamma \quad \Longrightarrow \quad 2\alpha + \beta - \gamma = 6$,

Consequently, the numbers α, β and γ must satisfy the given equations.

(c) If we take A to be the matrix in part (a) and the vectors

$$\boldsymbol{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix},$$

then the system of linear equations in part (b) can be written as $A\boldsymbol{x} = \boldsymbol{b}$. As such, since we found the inverse of A in part (b), we can use this to see that the required solution is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = -\frac{1}{6}\begin{pmatrix} -3 & 2 & -1 \\ 3 & -4 & -1 \\ -3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 6 \\ -30 \\ 18 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix},$$

and so we have $\alpha = -1$, $\beta = 5$ and $\gamma = -3$. Indeed, as the degree of homogeneity of the function in part (b) is given by

$$\text{degree of numerator} - \text{degree of denominator} = (1 + 2 + 3) - (6 + \gamma) = -\gamma,$$

we see that f is homogeneous of degree 3 if we use the values that we have just found.

Question 8

(a) We can write the given system of difference equations as

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} \quad \text{and so} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & \frac{5}{2} \end{pmatrix},$$

is the sought after 2×2 matrix.

To find the eigenvalues of this matrix, as in Sections 4.12–4.15 of the subject guide, we solve the equation

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ 1 & \frac{5}{2}-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (1-\lambda)(\tfrac{5}{2}-\lambda) - 1 = 0,$$

which, multiplying out the brackets, gives us the quadratic equation

$$\lambda^2 - \tfrac{7}{2}\lambda + \tfrac{3}{2} = 0 \quad \Longrightarrow \quad (\lambda - \frac{1}{2})(\lambda - 3) = 0,$$

and so the eigenvalues are $1/2$ and 3. To find the corresponding eigenvectors we seek a non-zero vector, $\boldsymbol{x}$, which is a solution to the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$, i.e.

- For $\lambda = 1/2$, we solve

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x + 2y = 0 \quad \Longrightarrow \quad x = -2y \quad \Longrightarrow \quad \boldsymbol{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

(or any non-zero multiple of this) is an eigenvector.

- For $\lambda = 3$, we solve

$$\begin{pmatrix} -2 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad -2x + y = 0 \quad \Longrightarrow \quad y = 2x \quad \Longrightarrow \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

(or any non-zero multiple of this) is an eigenvector.

Consequently, if we take

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix},$$

we have an invertible matrix, P, and a diagonal matrix, D, such that $P^{-1}AP = D$.[5]

(b) As in Section 5.6.2 of the subject guide, we can now use our answer to part (a) to solve the given coupled first-order differential equations. We let

$$\boldsymbol{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad \text{and} \quad \boldsymbol{u}_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

where $\boldsymbol{x}_t$ and $\boldsymbol{u}_t$ are related by $\boldsymbol{x}_t = P\boldsymbol{u}_t$. This means that $\boldsymbol{x}_{t-1} = P\boldsymbol{u}_{t-1}$ and, substituting this into $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$, we get

$$P\boldsymbol{u}_t = AP\boldsymbol{u}_{t-1} \quad \Longrightarrow \quad \boldsymbol{u}_t = P^{-1}AP\boldsymbol{u}_{t-1} \quad \Longrightarrow \quad \boldsymbol{u}_t = D\boldsymbol{u}_{t-1},$$

as $P^{-1}AP = D$. Using this, we have

[5] Of course, this is only one of the many possible pairs of matrices that we could choose for P and D: others are possible depending on which eigenvectors we choose when we form the columns of P and the order in which we choose to place them in P. For instance, choosing the other order for the eigenvectors we found above, we can see that

$$P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

is another possible answer here.

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u_{t-1} \\ v_{t-1} \end{pmatrix} \implies u_t = \tfrac{1}{2}u_{t-1} \quad \text{and} \quad v_t = 3v_{t-1},$$

and this pair of difference equations can easily be solved to get

$$u_t = A(\tfrac{1}{2})^t \qquad \text{and} \qquad v_t = B(3^t),$$

for arbitrary constants A and B. This means that, using $\boldsymbol{x}_t = P\boldsymbol{u}_t$, we have

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A(\frac{1}{2})^t \\ B(3^t) \end{pmatrix} = \begin{pmatrix} -2A(\frac{1}{2})^t + B(3^t) \\ A(\frac{1}{2})^t + 2B(3^t) \end{pmatrix},$$

which means that

$$x_t = -2A(\tfrac{1}{2})^t + B(3^t) \quad \text{and} \quad y_t = A(\tfrac{1}{2})^t + 2B(3^t),$$

is the general solution to our coupled system of difference equations.

(c) Using the initial conditions $x_0 = 2$ and $y_0 = 4$, our general solution from part (b) gives us the equations

$$2 = -2A + B \qquad \text{and} \qquad 4 = A + 2B,$$

which are easily solved to get $A = 0$ and $B = 2$. Consequently, we find that

$$x_t = 2(3^t) \qquad \text{and} \qquad y_t = 4(3^t),$$

is the required particular solution to our coupled system of difference equations. Indeed, as t increases, both of these sequences will *increase unboundedly*.[6]

(d) Returning to the general solution that we found in part (b), it should be clear that both x_t and y_t will tend to a finite limit as $t \to \infty$ if $B = 0$. To see how this is related to the values of x_0 and y_0, we note that

$$x_0 = -2A + B \qquad \text{and} \qquad y_0 = A + 2B,$$

and so, if $B = 0$, we must have

$$x_0 = -2A \qquad \text{and} \qquad y_0 = A,$$

which means that $x_0 = -2y_0$. Thus, the equation that x_0 and y_0 must satisfy in order for both x_t and y_t to have a finite limit as $t \to \infty$ is $x_0 + 2y_0 = 0$.

[6] Notice that this gives us a complete description of how these sequences behave as t increases. It is not sufficient to just say that they both tend to infinity as $t \to \infty$ nor is it sufficient to just say that they both increase as t increases.