Course information 2015–16
EC3120 Mathematical economics

Mathematical modelling is particularly helpful in analysing a number of aspects of economic theory. The course content includes a study of several mathematical models used in economics. Considerable emphasis is placed on the economic motivation and interpretation of the models discussed.

Prerequisite

If taken as part of a BSc degree, courses which must be passed before this course may be attempted:

EC2066 Microeconomics.
and either
MT105a Mathematics 1 **and** MT105b Mathematics 2
or
MT1174 Calculus

Aims and objectives

The course is specifically designed to:

- demonstrate to the student the importance of the use of mathematical techniques in theoretical economics
- enable the student to develop skills in mathematical modelling

Learning outcomes

At the end of this course and having completed the essential reading and activities students should be able to:

- ☑ use and explain the underlying principles, terminology, methods, techniques and conventions used in the subject
- ☑ solve economic problems using the mathematical methods described in the subject

Assessment

This course is assessed by a three hour unseen written examination.

Essential reading

For full details, please refer to the reading list

Dixit, Avinash K. *Optimization in Economics Theory*. (Oxford University Press)
Sydsæter, Knut, Peter Hammond, Atle Seierstad and Arne Strom *Further Mathematics for Economic Analysis*. (Pearson Prentice Hall)

Students should consult the *Programme Regulations for degrees and diplomas in Economics, Management, Finance and the Social Sciences* that are reviewed annually. Notice is also given in the *Regulations* of any courses which are being phased out and students are advised to check course availability.

Syllabus

This is a description of the material to be examined, as published in the *Programme handbook*. On registration, students will receive a detailed subject guide which provides a framework for covering the topics in the syllabus and directions to the essential reading.

Techniques of constrained optimisation. This is a rigorous treatment of the mathematical techniques used for solving constrained optimisation problems, which are basic tools of economic modelling. Topics include: Definitions of a feasible set and of a solution, sufficient conditions for the existence of a solution, maximum value function, shadow prices, Lagrangian and Kuhn Tucker necessity and sufficiency theorems with applications in economics, for example General Equilibrium theory, Arrow-Debreu securities and arbitrage.

Intertemporal optimisation. Bellman approach. Euler equations. Stationary infinite horizon problems. Continuous time dynamic optimisation (optimal control). Applications, such as habit formation, Ramsey-Kass-Coopmans model, Tobin's *q*, capital taxation in an open economy, are considered.

Tools for optimal control: ordinary differential equations. These are studied in detail and include linear 2nd order equations, phase portraits, solving linear systems, steady states and their stability.

Examiners' commentaries 2015

EC3120 Mathematical economics

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2014–15. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2014). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions – Zone A

Candidates should answer **EIGHT** of the following **TEN** questions: all **FIVE** from Section A (8 marks each) and any **THREE** from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Workings should be submitted for all questions requiring calculations. Any necessary assumptions introduced in answering a question are to be stated.

Section A

Answer all five questions from this section.

Question 1

State the envelope theorem and prove it for the simple case of an unconstrained maximisation problem.

Approaching the question

The envelope theorem can be stated as:

$$\frac{\mathrm{d}}{\mathrm{d}b} f(x(b), y(b);\, b) = \frac{\partial}{\partial b} L(x(b), y(b), \lambda(b);\, b).$$

We can prove this for the simple case of an unconstrainted problem. Let $f(x;\, a)$ be a continuous function of $x \in \mathbb{R}^n$ and the scalar a. For any a consider the problem of finding $\max f(x;\, a)$. Let $x^*(a)$ be the maximiser which we assume is differentiable with respect to a. We can show that:

$$\frac{\mathrm{d}}{\mathrm{d}a} f(x^*(a);\, a) = \frac{\partial}{\partial a} f(x^*(a);\, a).$$

Apply the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}a} f(x^*(a);\, a) = \sum_i \frac{\partial f}{\partial x_i}(x^*(a);\, a)\frac{\partial x_i^*}{\partial a}(a) + \frac{\partial f}{\partial a}(x^*(a);\, a).$$

Given that $(\partial f/\partial x_i)(x^*(a);\, a) = 0$ for all i by the first-order conditions, this gives:

$$\frac{\mathrm{d}}{\mathrm{d}a} f(x^*(a);\, a) = \frac{\partial}{\partial a} f(x^*(a);\, a).$$

Intuitively, when we are already at a maximum, changing slightly the parameters of the problem or the constraints does not affect the optimal solution.

Question 2

Consider the firm's cost minimisation problem.

(a) Define the cost function.

(b) State Shephard's Lemma for the cost function.

(c) Compute conditional factor demands and cost function for a Cobb–Douglas production function $f(K, L) = K^\alpha L^\beta$ with $\alpha + \beta = 1$, and verify that the cost function is homogeneous of degree one in input prices.

Approaching the question

(a) The cost function is the value function for the firm's cost minimisation problem. The cost function $c(w, r, q)$ is the minimum cost of producing output q:

$$c(w, r, q) = rK(r, w, q) + wL(r, w, q)$$

where $K(r, w, q)$ and $L(r, w, q)$ are conditional factor demands.

(b) Shepherd's Lemma for the cost function states that:

$$\begin{aligned} \frac{\partial c(r, w, q)}{\partial r} &= K(r, w, q) \\ \frac{\partial c(r, w, q)}{\partial w} &= L(r, w, q). \end{aligned}$$

(c) The cost minimisation problem for the firm is as follows:

$$\min_{L,K} rK + wL \quad \text{s.t. } K^\alpha L^\beta = q.$$

This can be restated as:

$$\max_{L,K} -rK - wL \quad \text{s.t. } K^\alpha L^\beta = q.$$

The Lagrangian is:

$$\mathcal{L} = -rK - wL + \lambda(K^\alpha L^\beta - q).$$

The first-order conditions give:

$$\frac{r}{w} = \frac{\lambda\alpha K^{\alpha-1}L^\beta}{\lambda\beta K^\alpha L^{\beta-1}} = \frac{\alpha L}{\beta K} \quad \Rightarrow L = \frac{\beta r}{\alpha w} K.$$

Substituting into the constraint:

$$\begin{aligned}
K^{\alpha}\left(\frac{\beta r}{\alpha w}K\right)^{\beta} &= q \\
\left(\frac{\beta r}{\alpha w}K\right)^{\beta}K^{\alpha+\beta} &= q \\
\left(\frac{\beta r}{\alpha w}K\right)^{\beta}K &= q \quad \Rightarrow K = \left(\frac{\alpha w}{\beta r}K\right)^{\beta} q \\
L &= \frac{\beta r}{\alpha w}K = \left(\frac{\beta r}{\alpha w}\right)^{1-\beta} q = \left(\frac{\beta r}{\alpha w}\right)^{\alpha} q.
\end{aligned}$$

Hence:

$$c(w, r, q) = r\left(\frac{\alpha w}{\beta r}\right)^{\beta} q + w\left(\frac{\beta r}{\alpha w}\right)^{\alpha} q.$$

Check that this is homogeneous of degree one in input prices:

$$\begin{aligned}
c(sw, sr, q) &= sr\left(\frac{\alpha sw}{\beta sr}\right)^{\beta} q + sw\left(\frac{\beta sr}{\alpha sw}\right)^{\alpha} q \\
&= sr\left(\frac{\alpha w}{\beta r}\right)^{\beta} q + sw\left(\frac{\beta r}{\alpha w}\right)^{\alpha} q \\
&= sc(w, r, q).
\end{aligned}$$

Question 3

Answer all parts of this question.

(a) Define a quadratic form.

(b) When do we call a symmetric matrix positive definite?

(c) Check the definiteness of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Approaching the question

(a) A quadratic form is a real-valued function:

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i\ leqj} a_{ij}x_ix_j$$

or, equivalently:

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

where A is a symmetric $n \times n$ matrix.

(b) We call A positive definite when $x^T Ax > 0$ for all $x \neq 0$ in $\mathbb{R}^n$. Equivalently, we can say that a matrix A is positive definite when all its n leading principal minors are strictly positive.

(c) Since $|A_1| = 1$ and $|A_2| = 4 - 6 = -1$, the matrix is indefinite.

Question 4

Solve the following ordinary differential equations:

(a) $y'' + 4y' - 5y = 0$**, where** $y(0) = 1$ **and** $y'(0) = 0$**.**

(b) $y'' + 4y' + 4y = 0$**, where** $y(0) = 1$ **and** $y'(0) = 0$**.**

Approaching the question

(a) The auxiliary equation is:

$$0 = r^2 + 4r - 5 = (r+5)(r-1)$$

so the general solution is:

$$y(t) = a\mathrm{e}^{-5t} + b\mathrm{e}^{t}.$$

The initial conditions give the equations $a + b = 1$ and $-5a + b = 0$. Subtracting these equations gives $6a = 1$, i.e. $a = 1/6$, and the first equation then yields $b = 5/6$.

(b) The auxiliary equation is:

$$0 = r^2 + 4r + 4 = (r+2)^2.$$

Thus the general solution, given coincident roots, is $y(t) = (a + bt)\mathrm{e}^{-2t}$. Setting $t = 0$ gives $a = 1$, i.e. $y(t) = (1 + bt)\mathrm{e}^{-2t}$. Hence:

$$y'(t) = (-2(1 + bt) + b)\mathrm{e}^{-2t}.$$

Then $y'(0) = 0$ implies $b = 2$.

Question 5

Consider the control problem

$$\max \int_0^\infty \mathrm{e}^{-rt} \left[x(t) - \frac{1}{2}u^2(t) \right] \, \mathrm{d}t$$

subject to $\dot{x} = u - x$ **and** $x(0) = x_0$**, where** $\dot{x}$ **denotes the time derivative.**

(a) Set up the Hamiltonian and state the necessary conditions for a maximum.

(b) Eliminate $u(t)$ **to obtain a system of first order linear differential equations in** x **and the co-state variable. Compute the family of solutions to this system explicitly.**

(c) Find the subfamily of solutions consistent with the optimality principle.

Approaching the question

Standard question: this is a slight variation on Example 9.1 in the subject guide.

Section B

Answer three questions from this section.

Question 6

Consider the utility function

$$u(x_1, x_2) = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$$

where x_1 **and** x_2 **are the quantities consumed of goods 1 and 2 respectively. Denote by** m **the consumer's income and by** u **the consumer's desired utility level; finally, denote by** p_1 **and** p_2 **the prices of good 1 and 2 respectively.**

(a) By solving the utility maximisation problem, find the uncompensated demand functions and the indirect utility.

(b) By using duality, find the expenditure function and compensated demands.

(c) State and verify Roy's identity.

Approaching the question

(a) Set up the utility maximisation problem:

$$\max_{x_1,x_2} \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$$

such that:

$$p_1x_1 + p_2x_2 = m.$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2 - \lambda(p_1x_1 + p_2x_2 - m).$$

The first-order conditions are:

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= \frac{1}{2x_1} - \lambda p_1 = 0\\ \frac{\partial L}{\partial x_2} &= \frac{1}{2x_2} - \lambda p_2 = 0\\ \frac{\partial L}{\partial \lambda} &= -(p_1x_1 + p_2x_2 - m) = 0.\end{aligned}$$

Hence:

$$\begin{aligned}\frac{1}{2x_1} &= \lambda p_1\\ \frac{1}{2x_2} &= \lambda p_2\\ p_1x_1 + p_2x_2 &= m.\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{1}{2p_1x_1} &= \frac{1}{2p_2x_2}\\ x_2 &= \frac{p_1}{p_2}x_1.\end{aligned}$$

Replacing in the budget constraint:

$$\begin{aligned}p_1x_1 + p_2\frac{p_1}{p_2}x_1 &= m\\ 2p_1x_1 &= m.\end{aligned}$$

Hence the uncompensated demand functions are:

$$\begin{aligned}x_1(p,m) &= \frac{m}{2p_1}\\ x_2(p,m) &= \frac{m}{2p_2}\end{aligned}$$

and the indirect utility function is:

$$v(p,m) = \frac{1}{2}\ln\frac{m}{2p_1} + \frac{1}{2}\ln\frac{m}{2p_2}.$$

(b) By duality:

$$v(p, e(p, u)) = u.$$

Here:

$$\begin{aligned} v(p, e(p, u)) &= \frac{1}{2} \ln \frac{e(p, u)}{2p_1} + \frac{1}{2} \ln \frac{e(p, u)}{2p_2} \\ &= \frac{1}{2} \left[\ln \frac{e(p, u)}{2p_1} + \ln \frac{e(p, u)}{2p_2} \right] \\ &= \frac{1}{2} \ln \frac{e(p, u)^2}{4p_1 p_2} \\ &= \ln \frac{e(p, u)}{2\sqrt{p_1 p_2}} \\ &= u. \end{aligned}$$

Hence the expenditure function is:

$$\frac{e(p, u)}{2\sqrt{p_1 p_2}} = \mathrm{e}^u \qquad \Rightarrow \qquad e(p, u) = 2\mathrm{e}^u \sqrt{p_1 p_2}.$$

We can now find the compensated demand functions by Shepherd's Lemma:

$$\begin{aligned} \frac{\partial e(p, u)}{\partial p_1} &= h_1(p, u) \\ \frac{\partial e(p, u)}{\partial p_2} &= h_2(p, u). \end{aligned}$$

Here:

$$\begin{aligned} h_1(p, u) &= \mathrm{e}^u \sqrt{\frac{p_2}{p_1}} \\ h_2(p, u) &= \mathrm{e}^u \sqrt{\frac{p_1}{p_2}}. \end{aligned}$$

(c) Roy's identity can be stated as follows:

$$x_i(p, m) = -\frac{\partial v(p, m)/\partial p_i}{\partial v(p, m)/\partial m}.$$

We can verify the identity for this case, where:

$$\begin{aligned} v(p, m) &= \frac{1}{2} \ln \frac{m}{2p_1} + \frac{1}{2} \ln \frac{m}{2p_2} \\ &= \frac{1}{2} \ln \frac{m^2}{4p_1 p_2} \\ &= -\frac{1}{2} \ln p_1 + \ln m + \frac{1}{2} \ln \frac{1}{4p_2}. \end{aligned}$$

We have:

$$\begin{aligned} \frac{\partial v(p, m)}{\partial p_1} &= -\frac{1}{2p_1} \\ \frac{\partial v(p, m)}{\partial m} &= \frac{1}{m}. \end{aligned}$$

By Roy's identity:

$$x_1(p, m) = \frac{m}{2p_1}.$$

Similarly, one can verify that:

$$x_2(p,m) = \frac{m}{2p_2}.$$

The uncompensated demand functions correspond to the ones we computed in (a), hence the identity is verified.

Question 7

Consider the utility maximisation problem with a quasi-linear utility function

$$\boldsymbol{u(x_1, x_2) = x_1^{1/2} + ax_2}$$

with $a > 0$. Denote by m the consumer's income and by p_1 and p_2 the prices of good 1 and 2, respectively.

(a) Write the utility maximisation problem as a constrained optimisation problem.

(b) Check if the Kuhn–Tucker theorem applies.

(c) Solve the problem by using the Kuhn–Tucker conditions.

Approaching the question

(a) We can write the utility maximisation problem as follows:

$$\max_{x_1,x_2} x_1^{1/2} + ax_2$$

such that:

$$\begin{aligned} p_1x_1 + p_2x_2 &\leq m \\ -x_1 &\leq 0 \\ -x_2 &\leq 0. \end{aligned}$$

(b) First note that the Kuhn–Tucker theorem applies when $u(x)$ is concave and differentiable and the constraint is convex and differentiable. Here the constraint is linear, hence convex; the objective function is concave when the Hessian matrix is negative semi-definite. Here the Hessian:

$$H = \begin{bmatrix} -x_1^{-1/2-1}/4 & 0 \\ 0 & 0 \end{bmatrix}$$

with $|H_1| < 0$ and $|H_2| = 0$, hence the Hessian is negative semi-definite and the objective function is concave. The Kuhn–Tucker theorem applies.

(c) The Lagrangian is:

$$\mathcal{L} = x_1^{1/2} + ax_2 - \lambda_0(p_1x_1 + p_2x_2 - m) - \lambda_1(-x_1) - \lambda_2(-x_2).$$

The Kuhn–Tucker conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{2}x_1^{-1/2} - \lambda_0 p_1 + \lambda_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= a - \lambda_0 p_2 + \lambda_2 = 0 \\ \lambda_0, \lambda_1, \lambda_2 &\geq 0 \\ \lambda_0(p_1x_1 + p_2x_2 - m) &= 0 \\ \lambda_1(-x_1) &= 0 \\ \lambda_2(-x_2) &= 0. \end{aligned}$$

Because the objective function is strictly increasing in x_1 and x_2 the budget constraint is binding, so $\lambda_0 > 0$ and $p_1x_1 + p_2x_2 = m$. Also, the constraint $-x_1 \leq 0$ cannot bind, for then the marginal utility with respect to x_1 would be infinitely large. Hence $x_1 > 0$ and $\lambda_1 = 0$. The other non-negativity constraint may or may not bind. Hence the Kuhn–Tucker conditions become:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{2}x_1^{-1/2} - \lambda_0 p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= a - \lambda_0 p_2 + \lambda_2 = 0 \\
\lambda_2 &\geq 0 \\
p_1x_1 + p_2x_2 &= m \\
\lambda_2(-x_2) &= 0.
\end{aligned}$$

Consider now the case $x_2 = 0$. The problem becomes:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{2}x_1^{-1/2} - \lambda_0 p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= a - \lambda_0 p_2 + \lambda_2 = 0 \\
p_1x_1 &= m.
\end{aligned}$$

From the budget constraint:

$$x_1 = \frac{m}{p_1}$$

and replacing in the first first-order condition we get:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2}x_1^{-1/2} - \lambda_0 p_1 = \frac{1}{2}\left(\frac{m}{p_1}\right)^{-1/2} - \lambda_0 p_1 = 0$$

from which we obtain:

$$\lambda_0 = \frac{1}{2p_1}\left(\frac{p_1}{m}\right)^{1/2} = \frac{1}{2\sqrt{p_1 m}}.$$

Replacing this value in the second first-order condition we obtain:

$$\frac{\partial \mathcal{L}}{\partial x_2} = a - \lambda_0 p_2 + \lambda_2 = a - \frac{1}{2\sqrt{p_1 m}}p_2 + \lambda_2 = 0$$

which gives:

$$\lambda_2 = \frac{p_2}{2\sqrt{p_1 m}} - a.$$

The solution:

$$\begin{aligned}
x_1 &= \frac{m}{p_1} \\
x_2 &= 0
\end{aligned}$$

is a valid solution when $\lambda_2 > 0$, hence when:

$$a < \frac{p_2}{2\sqrt{p_1 m}}.$$

For larger values of a, we have the interior solution with $x_1 > 0$ and $x_2 > 0$. The Kuhn–Tucker conditions become:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{2}x_1^{-1/2} - \lambda_0 p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= a - \lambda_0 p_2 = 0 \\
p_1x_1 + p_2x_2 &= m.
\end{aligned}$$

From the second first-order condition we obtain:

$$\lambda_0 = \frac{a}{p_2}$$

and replacing in the first first-order condition:

$$\frac{1}{2}x_1^{-1/2} - a\frac{p_1}{p_2} = 0$$

which gives:

$$x_1 = \frac{p_2^2}{4a^2p_1^2}.$$

Finally, we solve for x_2 by substituting in the budget constraint:

$$p_1\frac{p_2^2}{4a^2p_1^2} + p_2x_2 = m$$

which gives:

$$x_2 = \frac{m}{p_2} - \frac{p_2}{4a^2p_1}.$$

Notice that $x_2 > 0$ for $a > p_2/2\sqrt{p_1 m}$.

Hence the solution to the constrained optimisation problem is:

$$\text{for } a \leq \frac{p_2}{2\sqrt{p_1 m}}, \qquad x_1 = \frac{m}{p_1}, \qquad x_2 = 0$$

$$\text{for } a \geq \frac{p_2}{2\sqrt{p_1 m}}, \qquad x_1 = \frac{p_2^2}{4a^2p_1^2}, \qquad x_2 = \frac{m}{p_2} - \frac{p_2}{4a^2p_1}.$$

Question 8

Consider the following system of nonlinear equations:

$$\begin{aligned} \dot{x} &= x(4-4x) \\ \dot{y} &= y(2-x-y) \end{aligned}$$

where $\dot{x}$ and $\dot{y}$ denote time derivatives.

(a) Find all steady-states of the model.

(b) Linearise around each steady-state.

(c) Determine whether the steady-states are stable or unstable.

Approaching the question

(a) The steady states, i.e. those points where $\dot{x} = \dot{y} = 0$, are as follows.

i. $x = y = 0$, i.e. the point $(0, 0)$.

ii. $x = 0$ and $0 = 2 - x - y$, or $y = 2$, i.e. the point $(0, 2)$.

iii. $y = 0$ and $0 = 4 - 4x$, or $x = 1$, i.e. the point $(1, 0)$.

iv. $4 - 4x = 0$ and $2 - x - y = 0$, or $x = 1$ and $y = 1$, i.e. the point $(1, 1)$.

(b) The Taylor expansion around a steady state, i.e. any point (x_0, y_0) for which $F(x_0, y_0) = G(x_0, y_0) = 0$, is given by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J(x_0, y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \cdots$$

where the first derivative matrix:

$$J(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 4 - 8x_0 & 0 \\ -y_0 & 2 - x_0 - 2y_0 \end{pmatrix}.$$

For the four steady states derived in (a), we obtain the following linearisation matrices:

i. At $(0, 0)$:

$$\begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}.$$

ii. At $(0, 2)$:

$$\begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}.$$

iii. At $(1, 0)$:

$$\begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}.$$

iv. At $(1, 1)$:

$$\begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -1 & -1 \end{pmatrix}.$$

(c) We calculate the eigenvalues of $J(x_0, y_0)$ to determine the stability of the steady state.

i. The eigenvalues are 4 and 2, so it is unstable.

ii. The eigenvalues are 4 and -2, so it is a saddle point.

iii. The eigenvalues are -4 and 1, so it is a saddle point.

iv. The eigenvalues are -4 and -1, so it is locally stable.

Question 9

You decide to borrow £M over a period of T years via a mortgage with continuous repayment. It can be shown that your debt $D(t)$ satisfies the differential equation

$$\dot{D} - rD = -A$$

where $\dot{D}$ denotes the time derivative of debt, r is the constant interest rate paid and A is the rate of continuous repayment, together with the boundary conditions

$$D(0) = M \quad \text{and} \quad D(T) = 0.$$

(a) You may assume that the differential equation has the solution

$$D(t) = a + be^{rt}, \quad 0 \leq t \leq T.$$

Show that

$$a + b = M \quad \text{and} \quad a + be^{rT} = 0.$$

(b) Hence show that your mortgage debt is given by

$$D(t) = M\left(\frac{1 - e^{-r(T-t)}}{1 - e^{-rT}}\right)$$

and that the rate of continuous repayment A satisfies

$$A = \frac{Mr}{1 - e^{-rT}}$$

(c) Sketch the debt $D(t)$ as a function of time t, for $0 \leq t \leq T$, for (i) small interest rate r and (ii) large interest rate r. Provide a brief economic interpretation of your graphs.

Approaching the question

(a) The equations $D(0) = M$ and $D(T) = 0$ give $a + b = M$ and $a + be^{rT} = 0$, respectively.

(b) Solving these linear equations for a and b gives the stated expression for $D(t)$, after dividing fractions by e^{rT}. Substitution in the differential equation gives the stated value for A.

(c) If r is small, then the Taylor series $e^z \approx 1 + z$ gives:

$$D(t) \approx M\left(\frac{r(T-t)}{rT}\right) = M\left(1 - \frac{t}{T}\right).$$

Therefore, the debt's time evolution for small r is linear, reducing to zero at $t = T$: the repayments are reducing the initial debt and interest payments are negligible.

However, if r is large, then the fact that $e^{-x} \to 0$, as $x \to \infty$, implies that:

$$D(t) \approx M, \qquad \text{for } 0 \leq t \leq T.$$

Therefore, the debt is hardly reduced in the early stages of the mortgage when r is large: the repayments are mostly paying interest.

Question 10

The city of Dystopia is worried by its increasing pollution. Let c_t denote Dystopian consumption at time t, with corresponding pollution p_t. The Dystopian Government wishes to maximise social welfare in the following sense

$$\max \int_0^\infty e^{-\delta t}\left(u(c_t) - v(p_t)\right) \, dt$$

subject to $\dot{p}_t = \alpha c_t - \beta p_t - \gamma$, where $\dot{p}_t$ denotes the time derivative, p_0 is given, δ is the discount factor, and α, β and γ are positive constants. You may assume that u and v are differentiable functions; u is increasing and concave in c_t; v is increasing and convex in p_t.

(a) Identify the control variable and the state variable and briefly comment on the economic interpretation for the objective function and the constraint.

(b) State the Hamiltonian.

(c) Use the first order conditions to prove that $u'(c_t) = -\alpha\lambda_t e^{\delta t}$.

(d) Use the Hamiltonian system to show that optimum consumption satisfies the differential equation $u''(c_t)\dot{c}_t = (\beta + \delta)u'(c_t) - \alpha v'(p_t)$, where $\dot{c}_t$ denotes the time derivative of consumption.

Approaching the question

(a) The state variable is p_t and the control variable is c_t. Welfare is increasing with consumption but decreasing with pollution. There is a trade-off between consumption and pollution because an increase in consumption has a positive impact on pollution through the constraint, which in turn decreases welfare.

(b) The present value Hamiltonian can be written as:

$$H = e^{-\delta t}[u(c_t) - v(p_t)] + \lambda_t(\alpha c_t - \beta p_t - \gamma)$$

where λ_t is the co-state variable.

(c) The first-order condition is:

$$\frac{\partial H}{\partial c_t} = 0 \qquad \Leftrightarrow \qquad e^{-\delta t}u'(c_t) + \lambda_t \alpha = 0.$$

Hence we obtain:

$$u'(c_t) = -\alpha\lambda_t e^{\delta t}$$

as requested.

(d) We have:

$$\dot{\lambda}_t = -\frac{\partial H}{\partial p_t} \qquad \Leftrightarrow \qquad \dot{\lambda}_t = \mathrm{e}^{-\delta t} v'(p_t) + \lambda_t \beta$$

$$\dot{p}_t = +\frac{\partial H}{\partial \lambda_t} \qquad \Leftrightarrow \qquad \dot{p}_t = \alpha c_t - \beta p_t - \gamma.$$

Also, the transversality condition:

$$\lim_{t \to \infty} \lambda_t = 0.$$

Recall that:

$$u'(c_t) = -\alpha \lambda_t \mathrm{e}^{\delta t}.$$

We differentiate with respect to time, to obtain:

$$u''(c_t)\dot{c}_t = -\alpha \dot{\lambda}_t \mathrm{e}^{\delta t} - \alpha \lambda_t \delta \mathrm{e}^{\delta t}.$$

Recall that:

$$\begin{aligned} \dot{\lambda}_t &= \mathrm{e}^{-\delta t} v'(p_t) + \lambda_t \beta \\ \lambda_t &= -\frac{u'(c_t)}{\alpha} \mathrm{e}^{-\delta t}. \end{aligned}$$

By substituting, we obtain:

$$\begin{aligned} u''(c_t)\dot{c}_t &= -\alpha \left[\mathrm{e}^{-\delta t} v'(p_t) + \lambda_t \beta\right] \mathrm{e}^{\delta t} - \alpha \lambda_t \delta \mathrm{e}^{\delta t} \\ &= -\alpha v'(p_t) - \alpha \lambda_t \beta \mathrm{e}^{\delta t} - \alpha \lambda_t \delta \mathrm{e}^{\delta t} \\ &= -\alpha v'(p_t) - \alpha \lambda_t (\beta + \delta) \mathrm{e}^{\delta t} \\ &= -\alpha v'(p_t) + a \frac{u'(c_t)}{a} \mathrm{e}^{-\delta t} (\beta + \delta) \mathrm{e}^{\delta t} \\ &= (\beta + \delta) u'(c_t) - \alpha v'(p_t). \end{aligned}$$

Examiners' commentaries 2015

EC3120 Mathematical economics

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2014–15. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2014). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions – Zone B

Candidates should answer **EIGHT** of the following **TEN** questions: all **FIVE** from Section A (8 marks each) and any **THREE** from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Workings should be submitted for all questions requiring calculations. Any necessary assumptions introduced in answering a question are to be stated.

Section A

Answer all five questions from this section.

Question 1

Answer all parts of this question.

(a) Define increasing, decreasing and constant returns to scale.

(b) Show that a Cobb–Douglas production function $f(K, L) = K^{\alpha} L^{\beta}$ has constant returns to scale if $\alpha + \beta = 1$.

(c) Prove that a concave production function $f(K, L)$ with $f(0, 0) = 0$ has decreasing or constant returns to scale.

Approaching the question

(a) A production function $f(K, L)$ has increasing returns to scale if for any $s > 1$, $f(sK, sL) > sf(K, L)$; it has constant returns to scale if for any $s > 1$,