UNIVERSITY OF LONDON | INTERNATIONAL PROGRAMMES

Course information 2015–16
MT3095 Further mathematics for economists

This course provides students with the mathematical techniques and methods which find application in economics and related areas, and enables students to understand why, and in what circumstances, these techniques work.

Prerequisite

If taken as part of a BSc degree, courses which must be passed before this course may be attempted:

Either both
MT105a Mathematics 1 ***and*** 05b Mathematics 2
or
MT1174 Calculus

Exclusion

May not be taken with:
MT2116 Abstract mathematics ***or***
MT2176 Further calculus ***or***
MT2175 Further linear algebra

Aims and objectives

The course is designed to:

- enable students to acquire skills in further methods of calculus and linear algebra, as required for their use in advanced economics-based subjects
- enable students to understand the underlying theory behind these techniques and those of more basic mathematics courses (such as 05a Mathematics 1 and 05b Mathematics 2)
- prepare students for advanced study in theoretical aspects of economics-based subjects.

Assessment

This course is assessed by a three hour unseen written examination.

Learning outcomes

At the end of this course and having completed the essential reading and activities students should be able to:

- ☑ use the concepts, terminology, methods and conventions covered in the unit to solve mathematical problems in this subject.
- ☑ demonstrate an understanding of the underlying principles of the subject.
- ☑ solve unseen mathematical problems involving understanding of these concepts and application of these methods.
- ☑ prove statements and to formulate precise mathematical arguments.

Recommended reading

For full details please refer to the reading list.

Most topics in this subject are covered in great detail by many texts. For this reason we do not specify essential reading for this course. However, textbook reading is essential to provide more in-depth explanation and many examples to study and exercises to work through. Listed **in order of usefulness**, rather than alphabetically, the first three we recommend are:

Simon, C.P. and L. Blume *Mathematics for Economists.* (New York and London: W.W. Norton and Company)
Anton, Howard A. *Elementary Linear Algebra.* (Wiley Text Books)
Ostaszewski, A. *Advanced Mathematical Methods.* (Cambridge: Cambridge University Press)

Students should consult the *Programme Regulations for degrees and diplomas in Economics, Management, Finance and the Social Sciences* that are reviewed annually. Notice is also given in the *Regulations* of any courses which are being phased out and students are advised to check course availability.

Syllabus

This is a description of the material to be examined, as published in the *Programme handbook*. On registration, students will receive a detailed subject guide which provides a framework for covering the topics in the syllabus and directions to the essential reading.

Linear algebra: Vector spaces, linear independence and dependence, bases and dimension, rank and nullity of a matrix. Linear mappings, their rank and nullity, their matrix representation, and change of basis. Eigenvalues and eigenvectors. Diagonalisation of matrices, with applications to systems of difference and differential equations (including stability). Quadratic forms and orthogonal diagonalisation. Inner product spaces, norms, orthogonality and orthonormalisation.

Functions and mathematical analysis: Sets and functions. Supremum and infinum of bounded sets. Limits of sequences in R and R^m. Limits and continuity of functions.
Open subsets and closed subsets of R^m.
Compact subsets of R^m. Convex sets, convex and concave funstions. Gradients and directional derivatives. The Jacobian derivative. The Edgeworth Box and contract curves.

Optimisation: Inconstrained optimisation and the second-order conditions. Constrained optimisation and the Kuhn-Tucker theorem. Envelope Theorems. Theory of linear programming (computational methods will not be included). Duality, with applications. Basic Game Theory.

Note: Candidates will be expected to work with formal definitions and be able to prove results as well as apply techniques and methods.

Examiners' commentaries 2015

MT3095 Further mathematics for economists

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2014–15. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refers to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements — if none are available, please use the contents list and index of the new edition to find the relevant section.

Specific comments on questions — Zone A

Question 1

For relevant reading, see Chapters 2 and 3 of the subject guide.

The *null space*, $N(A)$, of a matrix A is the set of all vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{0}$.

Using row operations on the matrix A, we see that

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 4\\ 1 & 3 & -3\\ 3 & 1 & 7\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 1\\ 0 & -1 & 2\\ 0 & 2 & -4\\ 0 & -2 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 3\\ 0 & 1 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

is its reduced echelon form. Indeed, as this has two non-zero rows, we can see that the rank of A is two.

Using the reduced echelon form, we can solve the matrix equation $A\boldsymbol{x} = \boldsymbol{0}$ to see that, with $z = t \in \mathbb{R}$, we have

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}-3t\\ 2t\\ t\end{pmatrix} = t\begin{pmatrix}-3\\ 2\\ 1\end{pmatrix} \quad \text{which means that} \quad \left\{\begin{pmatrix}-3\\ 2\\ 1\end{pmatrix}\right\},$$

is a basis of the solution space of this equation.

The columns of A, let's call them $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and $\boldsymbol{c}_3$ respectively, do *not* form a linearly independent set of vectors as, for instance, we can use the solution above to see that they satisfy the linear dependence relation $-3\boldsymbol{c}_1 + 2\boldsymbol{c}_2 + \boldsymbol{c}_3 = \boldsymbol{0}$.

Question 2

For relevant reading, see Chapter 9 of the subject guide.

Given the surfaces $f(x,y,z)=0$ and $g(x,y,z)=1$ where

$$f(x,y,z)=x^2(y-z)+y^3 \quad \text{and} \quad g(x,y,z)=x^2+y^2+z^2,$$

we can see that, as $f(1,0,0)=0$ and $g(1,0,0)=1$, these two surfaces must certainly intersect at the point $(1,0,0)$. To see that they intersect orthogonally, we note that their gradient vectors are

$$\nabla f=\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}=\begin{pmatrix} 2x(y-z) \\ x^2+3y^2 \\ -x^2 \end{pmatrix} \quad \text{and} \quad \nabla g=\begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}=\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

which means that, evaluating these at $(1,0,0)$ and taking their inner product, we get

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\rangle = 0+0+0=0,$$

and so they are certainly orthogonal at this point.

A unit vector in the direction of the vector $\boldsymbol{u}=(1,0,-1)^T$ is

$$\hat{\boldsymbol{u}}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{as} \quad \|\boldsymbol{u}\|^2=1+0+1=2,$$

which means that the rate of change of f at $(1,0,0)$ in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}}f(1,0,0)=\left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = \frac{0+0+1}{\sqrt{2}}=\frac{1}{\sqrt{2}},$$

and the corresponding rate of change of g is

$$D_{\boldsymbol{u}}g(1,0,0)=\left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = \frac{2+0+0}{\sqrt{2}}=\frac{2}{\sqrt{2}}.$$

Thus, as the latter is greater than the former, g has the greater rate of increase at the point $(1,0,0)$ in the direction $\boldsymbol{u}$.

The Cartesian equation of the tangent plane to the surface $f(x,y,z)=0$ at the point (a,b,c) is given by

$$\left\langle \begin{pmatrix} 2a(b-c) \\ a^2+3b^2 \\ -a^2 \end{pmatrix}, \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} \right\rangle = 0 \quad \Longrightarrow \quad 2a(b-c)(x-a)+(a^2+3b^2)(y-b)-a^2(z-c)=0.$$

For this tangent plane to go through the origin, the point $(0,0,0)$ must satisfy this equation and it does as

$$2a(b-c)(0-a)+(a^2+3b^2)(0-b)-a^2(0-c)=-3[a^2b-a^2c+b^3]=-3[a^2(b-c)+b^3]=0,$$

because the point (a,b,c) is on the surface $f(x,y,z)=0$ and so it must satisfy the equation of that surface.

Question 3

For relevant reading, see Chapter 10 of the subject guide.

The function

$$g(x, y, z) = x^2 - xz + y^2 + z^2 - 2x - 3y,$$

has stationary points when its first-order partial derivatives, i.e.

$$f_x = 2x - z - 2, \quad f_y = 2y - 3 \quad \text{and} \quad f_z = -x + 2z,$$

are all equal to zero and so we start by solving the equations $f_x = 0$, $f_y = 0$ and $f_z = 0$, i.e.

$$2x - z = 2, \quad 2y = 3 \quad \text{and} \quad x = 2z,$$

simultaneously. Here, the second equation gives us $y = 3/2$ and, substituting the third equation in to the first equation we get $3z = 2$ so that $z = 2/3$ and $x = 4/3$. Thus, this function has one stationary point, namely $(4/3, 3/2, 2/3)$.

To classify this stationary point, we see that the Hessian matrix is

$$D^2 f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix},$$

and so, at our stationary point (and, indeed, at every other point), we see that its principal minors are

$$2 > 0, \quad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \text{and} \quad \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 2(4-1) = 6 > 0,$$

if we do a cofactor expansion along row 2. As the principal minors are all positive, $D^2 f$ is positive definite at our stationary point and so it is a local minimum.

Indeed, as observed above, $D^2 f$ is positive definite at *every* point in $\mathbb{R}^3$ and so our stationary point is a global minimum of f because this function is convex. There is, of course, no global maximum for this function because we can see that, say, $f(0, 0, z) = z^2 \to \infty$ as $z \to \infty$.

Question 4

For relevant reading, see Section 10.3 of the subject guide.

In standard form, our inequality constraint optimisation problem becomes

$$\begin{aligned} \text{maximise} \quad & f(x, y) = -\left(x + \frac{1}{2}\right)^2 - \frac{y^2}{2}, \\ \text{subject to} \quad & g_1(x, y) = e^{-x} - y \leq 0, \\ & g_2(x, y) = 2y - 1 \leq 0. \end{aligned}$$

The Jacobian of both constraint functions is

$$\begin{pmatrix} -e^{-x} & -1 \\ 0 & 2 \end{pmatrix},$$

and, since $e^{-x} \neq 0$ for all $x \in \mathbb{R}$, this always has full rank. Thus, the NDCQ holds.

As there are no non-negativity constraints on x and y we use the normal Lagrangian (see Subsection 10.3.2 of the subject guide), which is

$$L(x, y, \lambda_1, \lambda_2) = -\left(x + \frac{1}{2}\right)^2 - \frac{y^2}{2} - \lambda_1(e^{-x} - y) - \lambda_2(2y - 1).$$

The equations to be solved are therefore:

$$0 = -2\left(x + \frac{1}{2}\right) + \lambda_1 e^{-x}, \tag{4.1}$$

$$0 = -y + \lambda_1 - 2\lambda_2, \tag{4.2}$$

$$0 = \lambda_1\left(e^{-x} - y\right), \tag{4.3}$$

$$0 = \lambda_2\left(2y - 1\right), \tag{4.4}$$

$$\lambda_1, \lambda_2 \geq 0, \tag{4.5}$$

$$e^{-x} \leq y, \tag{4.6}$$

$$2y \leq 1. \tag{4.7}$$

For equation (4.3) to be satisfied, we must have either $\lambda_1 = 0$ or $e^{-x} = y$, while equation (4.4) yields either $\lambda_2 = 0$ or $2y = 1$. These combine to give 4 possible cases to consider:

- <u>$\lambda_1 = 0$ and $\lambda_2 = 0$</u>

  In this case our first order conditions, equations (4.1) and (4.2), yield $x = -1/2$ and $y = 0$ respectively.

  However, if $y = 0$, then our first inequality constraint, equation (4.6), says $e^{-x} \leq 0$, which is not possible for any value of $x \in \mathbb{R}$.

  Thus this case yields no candidates for the maximum.

- <u>$\lambda_1 = 0$ and $2y = 1$</u>

  In this case, equation (4.2) says that $\lambda_2 = -1/4 < 0$, which violates the non-negativity requirement of equation (4.5). (Equation (4.6) would also say $e^{-1/2} \leq 1/2$, which also turns out to be incorrect but is less obvious.)

  Thus this case yields no candidates for the maximum.

- <u>$y = e^{-x}$ and $\lambda_2 = 0$</u>

  Equation (4.2) yields $y = \lambda_1 = e^{-x}$. Equation (4.1) then becomes

  $$2x + 1 = e^{-2x},$$

  where inspection yields the solution $x = 0$.

  Equation (4.2) then implies that $y = 1$. However, this violates our second inequality constraint, equation (4.7).

  Thus this case yields no candidates for the maximum.

- <u>$y = e^{-x}$ and $2y = 1$</u>

  This case yields the only solution to these equations:

  $$(x^\star, y^\star, \lambda_1^\star, \lambda_2^\star) = \left(\ln 2,\ \frac{1}{2},\ 2 + 4\ln 2,\ \frac{3}{4} + 2\ln 2\right). \tag{4.8}$$

As we have found only one candidate for the maximum, and we have concluded that the NDCQ holds everywhere, we therefore conclude that the solution to our maximisation problem is $x = \ln 2$ and $y = 1/2$.

Question 5

For relevant reading, see Chapter 8 of the subject guide.

The sequence (x_n) of real numbers *converges* to the real number L as n tends to infinity if and only if, for any $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |x_n - L| < \varepsilon.$$

To show that the given sequence converges to 2 as n tends to infinity, we must therefore show that we can make $|x_n - 2|$ smaller than any arbitrary value $\varepsilon > 0$ for all $n > N$, for some suitable $N \in \mathbb{N}$ that we must determine.

$$|x_n - 2| = \left| \frac{4n^4 - n^3}{2n^4 + 2n^2 + 1} - 2 \right| = \left| \frac{-n^3 - 4n^2 - 2}{2n^4 + 2n^2 + 1} \right|.$$

Before we can start to introduce inequalities, we *must* remove the absolute value signs.[1] In this case, the only concerns are the negative signs in the numerator (as the denominator is always positive). However:

$$\left|-n^3 - 4n^2 - 2\right| = \left|-(n^3 + 4n^2 + 2)\right| = n^3 + 4n^2 + 2.$$

Thus

$$|x_n - 2| = \frac{n^3 + 4n^2 + 2}{2n^4 + 2n^2 + 1} < \frac{n^3 + 4n^3 + 2n^3}{2n^4} = \frac{7n^3}{2n^4} = \frac{7}{2n}.$$

We therefore choose $N \in \mathbb{N}$ such that $N > 7/(2\varepsilon)$, so that

$$n > N > \frac{7}{2\varepsilon} \quad \Rightarrow \quad \frac{7}{2n} < \frac{7}{2N} < \varepsilon \quad \Rightarrow \quad |x_n - 2| < \varepsilon.$$

Question 6

For relevant reading, see Chapter 11 of the subject guide.

Writing the linear program in standard form, we have

$$\text{minimise} \quad (8\ ,\ 5\ ,\ -6\ ,\ -1) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 2 & 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is, minimise $\boldsymbol{b}^T\boldsymbol{y}$ subject to $A^T\boldsymbol{y} \geq \boldsymbol{c}$ and $\boldsymbol{y} \geq \boldsymbol{0}$, where

$$A^T = \begin{pmatrix} 1 & 1 & -4 & 1 \\ 2 & 1 & -2 & -2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 8 \\ 5 \\ -6 \\ -1 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

[1] Many candidates did not do this and therefore lost marks for writing down statements that were incorrect.

The dual problem to our minimisation problem is defined to be the following *maximisation* problem:

$$\text{maximise} \qquad \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{subject to} \qquad A\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}.$$

That is

$$\text{maximise} \quad (5\ ,\ 6)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ -4 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 5 \\ -6 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The *six* inequalities above describe six half-planes in $\mathbb{R}^2$, and the intersection of these forms the feasible set for our dual problem (see Figure 1). The extreme points of this feasible set are $(0, 3)$, $(0, 4)$, $(2, 3)$, $(3, 2)$ and $(1, 1)$.

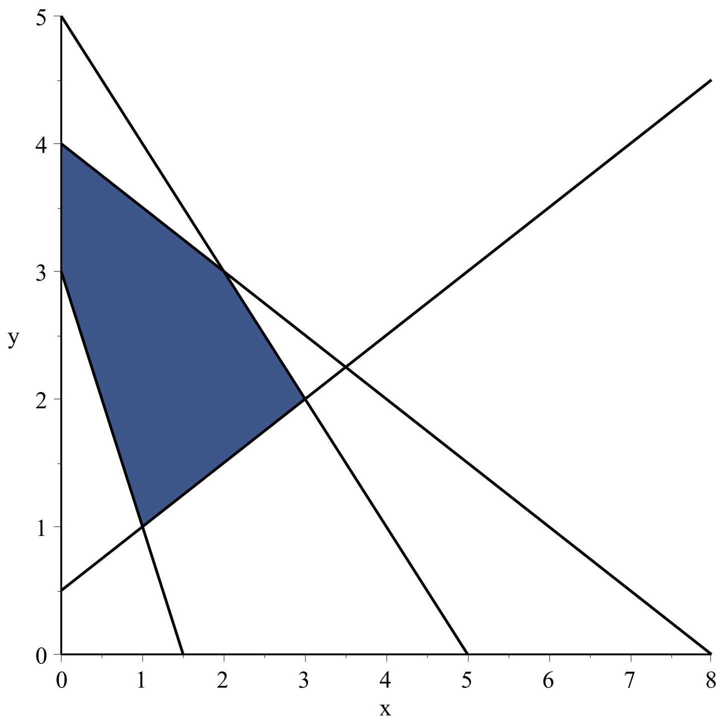

Figure 1: The feasible set for dual linear program in Question 6.

The optimal value of the problem must occur at one of the extreme points of the feasible set (see Theorem 11.7 of the subject guide). Evaluating the objective function $\boldsymbol{c}^T\boldsymbol{x}$ at the points identified above, we find that the maximum occurs at the point $(2, 3)$, with value 28. By the duality theorem, the value of the original minimisation problem is also 28.

Question 7

For relevant reading, see Chapters 5 and 6 of the subject guide.

(a) A vector, $\boldsymbol{v} \neq \mathbf{0}$, is an *eigenvector* of a matrix, A, if $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for some $\lambda \in \mathbb{R}$.

Using this definition, we then see that

$$A\boldsymbol{v} = \begin{pmatrix} 7 & 0 & -3 \\ 1 & 6 & 5 \\ 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -14 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix} = 2\boldsymbol{v},$$

and so, $\boldsymbol{v}$ is an eigenvector of A with a corresponding eigenvalue of two.

(b) To find the other eigenvalues of A we solve its characteristic equation $|A - \lambda I| = 0$ which, in this case, is given by

$$\begin{vmatrix} 7-\lambda & 0 & -3 \\ 1 & 6-\lambda & 5 \\ 5 & 0 & -1-\lambda \end{vmatrix} = 0,$$

and so, using a cofactor expansion down (say) the second column, this gives us

$$(6-\lambda)[(7-\lambda)(-1-\lambda)+15] = 0 \implies (6-\lambda)[\lambda^2 - 6\lambda + 8] = 0 \implies (6-\lambda)(\lambda-2)(\lambda-4) = 0.$$

Thus, along with $\lambda = 2$, the other eigenvalues of A are $\lambda = 4$ and $\lambda = 6$. We can then find the eigenvectors corresponding to these other eigenvectors by solving the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. This gives us

- For $\lambda = 4$, we solve $(A - 4I)\boldsymbol{x} = \mathbf{0}$ by noting that

$$A - 4I = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 5 \\ 5 & 0 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \implies \boldsymbol{x} = t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

- For $\lambda = 6$, we solve $(A - 6I)\boldsymbol{x} = \mathbf{0}$ by noting that

$$A - 6I = \begin{pmatrix} 1 & 0 & -3 \\ 1 & 0 & 5 \\ 5 & 0 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \boldsymbol{x} = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

Then, with the eigenvector for $\lambda = 2$ from part (a), we can take

$$P = \begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

as the invertible matrix P and the diagonal matrix D such that $P^{-1}AP = D$.

(c) The given system of linear recurrence equations can be written as $\boldsymbol{x}_t = A\boldsymbol{x}_{t-1}$ with $\boldsymbol{x}_t = (x_t, y_t, z_t)^T$. So, using this equation recursively and $A = PDP^{-1}$ from part (b), we see that its solution will then be given by

$$\boldsymbol{x}_t = A^t\boldsymbol{x}_0 = PD^tP^{-1}\boldsymbol{x}_0 = PD^t\boldsymbol{u},$$

where $\boldsymbol{u} = P^{-1}\boldsymbol{x}_0 = (A, B, C)^T$ is a vector of arbitrary constants. Thus, we have

$$\boldsymbol{x}_t = P\begin{pmatrix} 4^t & 0 & 0 \\ 0 & 6^t & 0 \\ 0 & 0 & 2^t \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ -3 & 1 & -7 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} A(4^t) \\ B(6^t) \\ C(2^t) \end{pmatrix} = \begin{pmatrix} A(4^t) + 3C(2^t) \\ -3A(4^t) + B(6^t) - 7C(2^t) \\ A(4^t) + 5C(2^t) \end{pmatrix},$$

as the general solution.

To find the required particular solutions, we use the given initial conditions to get the equations

$$A+3C=-1, \quad -3A+B-7C=2 \quad \text{and} \quad A+5C=1.$$

Subtracting the first of these from the third gives us $2C=2$ so that $C=1$. The first equation then gives us $A=-4$ and, putting these values into the second equation, we then find that $B=2+3(-4)+7(1)=-3$. Thus, we have

$$x_t=-4(4^t)+3(2^t), \quad y_t=12(4^t)-3(6^t)-7(2^t) \quad \text{and} \quad z_t=-4(4^t)+5(2^t),$$

as the answers.

Question 8

(a) For relevant reading, see Section 9.6 of the subject guide.

With $u_R(x,y)=x^{1/3}y$ and $u_M(x,y)=x^2y^{1/2}$, the contract curve is found by solving

$$\nabla u_R=\lambda\nabla v_M,$$

where $v_M(x,y)=u_M(20-x,10-y)=(20-x)^2(10-y)^{1/2}$ as the amounts of wheat and oats available are 20 and 10 bushels respectively. So, we have

$$\nabla u_A=\begin{pmatrix}\frac{1}{3}x^{-2/3}y\\ x^{1/3}\end{pmatrix} \quad \text{and} \quad \nabla v_B=\begin{pmatrix}-2(20-x)(10-y)^{1/2}\\ -\frac{1}{2}(20-x)^2(10-y)^{-1/2}\end{pmatrix},$$

which means that, eliminating λ from $\nabla u_A=\lambda\nabla v_B$, we find that

$$\frac{\frac{1}{3}x^{-2/3}y}{-2(20-x)(10-y)^{1/2}}=\frac{x^{1/3}}{-\frac{1}{2}(20-x)^2(10-y)^{-1/2}}.$$

Tidying this up, we get

$$\frac{20-x}{10-y}=12\left(\frac{x}{y}\right) \quad \Longrightarrow \quad 20y+11xy-120x=0,$$

as the equation of their contract curve.

(b) For relevant reading, see Chapter 11 of the subject guide.

In matrix/vector form, the primal problem is

$$\text{minimise} \quad (2\ ,\ -3\ ,\ 4)\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

subject to

$$\begin{pmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\geq\begin{pmatrix}-2\\ 3\\ -4\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\geq\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

That is, minimise $\boldsymbol{b}^T\boldsymbol{y}$ subject to $A^T\boldsymbol{y}\geq\boldsymbol{c}$ and $\boldsymbol{y}\geq\boldsymbol{0}$, where

$$A^T=\begin{pmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{pmatrix}, \quad \boldsymbol{b}=\begin{pmatrix}2\\ -3\\ 4\end{pmatrix}, \quad \boldsymbol{c}=\begin{pmatrix}-2\\ 3\\ -4\end{pmatrix} \quad \text{and} \quad \boldsymbol{y}=\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}.$$

The dual problem to our minimisation problem is defined to be the following *maximisation* problem:

$$\text{maximise} \qquad \boldsymbol{c}^T\boldsymbol{x}$$

$$\text{subject to} \qquad A\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}.$$

That is

$$\text{maximise} \quad (-2\ ,\ 3\ ,\ -4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve these problems *without* performing any calculations, we first show that both problems have *exactly the same domain.* As $A^T = -A$ and $\boldsymbol{c} = -\boldsymbol{b}$, the domain of the dual problem can be rewritten as

$$\begin{matrix} A\boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} -A\boldsymbol{x} \geq -\boldsymbol{b} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} A^T\boldsymbol{x} \geq \boldsymbol{c} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{matrix}$$

The result follows by simply renaming $\boldsymbol{x}$ as $\boldsymbol{y}$ in the last step.

Moreover, the objective function of the dual problem is the negative of that in the primal problem.

Because of *both* these facts, if the minimum of the original problem is V then the maximum of the dual must be $-V$.[2] However, by the strong law of duality, we know that these values must be equal: i.e. $V = -V$. Thus $V = 0$.

(c) The Intermediate Value Theorem is covered in Section 8.10 of the subject guide.

The **Intermediate Theorem** (see Theorem 8.16 in the subject guide) states that:

> Let f be a continuous function on $[a, b]$ and let K be such that $f(a) < K < f(b)$.
> Then, for some $c \in (a, b)$, $f(c) = K$.

Let $K \in \mathbb{R}$ be any real number. We must show that there is some $c \in \mathbb{R}$ such that $f(c) = K$. Since K is any real number, this will show that $\{f(x) \mid x \in \mathbb{R}\} = \mathbb{R}$.

Because f is not bounded above or below, there will be $x, y \in \mathbb{R}$ with $f(x) < K$ and $f(y) > K$. Thus $f(x) < K < f(y)$ and, by the Intermediate Value Theorem, there is therefore some c between x and y with $f(c) = K$.

Question 9

(a) For relevant reading, see Chapters 5 and 6 of the subject guide.

We write $f(x, y) = 3x^2 + 2xy + 3y^2$ as $\boldsymbol{x}^T A\boldsymbol{x}$ with $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

[2] Note, we can only conclude this if the domains are the same. Why?

To orthogonally diagonalise this matrix we start by finding its eigenvalues by solving its characteristic equation. This is given by

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad (3-\lambda)^2 - 1 = 0 \quad \Longrightarrow \quad \lambda = 3 \pm 1,$$

and so the eigenvalues of A are $\lambda = 4$ and $\lambda = 2$. We then have to find the corresponding eigenvectors by solving the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ for each eigenvalue.

- For $\lambda = 4$, we solve $(A - 4I)\boldsymbol{x} = \mathbf{0}$ by noting that

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x - y = 0 \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix},$$

  for $y \in \mathbb{R}$. As such, we can take $(1, 1)^T$ to be the required eigenvector.
- For $\lambda = 2$, we solve $(A - 2I)\boldsymbol{x} = \mathbf{0}$ by noting that

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad x + y = 0 \quad \Longrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix},$$

  for $y \in \mathbb{R}$. As such, we can take $(-1, 1)^T$ to be the required eigenvector.

We observe that these eigenvectors are already orthogonal and so, to get an orthogonal matrix P, we only need to make them unit vectors. Thus, as they both have a length of $\sqrt{1+1} = \sqrt{2}$, we can take

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

as the orthogonal matrix P and the diagonal matrix D such that $P^T AP = D$.

We now let $\boldsymbol{x} = P\boldsymbol{v}$ with $\boldsymbol{v} = (v, w)^T$ so that $\boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{v}^T P^T AP\boldsymbol{v} = \boldsymbol{v}^T D\boldsymbol{v}$ and this gives us

$$f(x, y) = \boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{v}^T D\boldsymbol{v} = 4v^2 + 2w^2.$$

Indeed, as P is orthogonal, $\boldsymbol{x} = P\boldsymbol{v}$ gives us $\boldsymbol{v} = P^T\boldsymbol{x}$ and so

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

allows us to express the coordinates v and w as linear functions of x and y.

(b) For relevant reading, see Chapter 8 of the subject guide.

(i) The **Bolzano-Weierstrass Theorem** (Theorem 8.25 of the subject guide) states that:

> Let A be a compact subset of $\mathbb{R}^n$ and let $(\boldsymbol{x}_n)$ be any sequence of points of A.
> Then $(\boldsymbol{x}_n)$ has a convergent subsequence whose limit is in A.

(ii) As A is compact, it is therefore bounded above. As A is non-empty and bounded above, the **Continuum property** states that A must have a least upper bound, or supremum, which we shall denote by $\sup A$.

(iii) We shall prove this by construction.

Given that $\sup A$ exists then,[3] for every $n \in \mathbb{N}$, we can find an $x_n \in A$ such that

$$\sup A - \frac{1}{n} < x_n \leq \sup A \qquad \Leftrightarrow \qquad |x_n - \sup A| < \frac{1}{n}.$$

Thus, by construction, (x_n) is a sequence of points in A that converges to the supremum of A.

[3]Here, we are using **Theorem 8.1** of the subject guide.

(iv) As (x_n) is a sequence in the set A, and A is a compact set, then the **Bolzano-Weierstrass Theorem** tells us that (x_n) must have a convergent subsequence (x_{n_k}) with limit $x \in A$.

Moreover, as the parent sequence (x_n) is also convergent with limit $\sup A$, this convergent subsequence must also converge to $\sup A$. Thus $\sup A = x \in A$.

Question 10

(a) For relevant reading, see Section 8.7 of the subject guide.

The **Sandwich Theorem** (see Theorem 8.10 in the subject guide) states that:

Let (a_n), (b_n), (c_n) be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $\lim\limits_{n\to\infty} b_n = L$.

Since $0 \leq 2^{-i} \leq 1$ for all $i = 1, 2, \ldots, n$, we have

$$0 \leq x_n \leq \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n},$$

where

$$\frac{1}{n^2+n} \leq \cdots \leq \frac{1}{n^2+2} \leq \frac{1}{n^2+1},$$

so that

$$0 \leq x_n \leq n \times \frac{1}{n^2+1}$$

for all $n \in \mathbb{N}$.

Here, using the **'Heredity' results** (see Theorem 8.8 in the subject guide), we have

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = \frac{0}{1+0} = 0,$$

and clearly $0 \to 0$. Hence, by the **Sandwich Theorem**, $\lim\limits_{n\to\infty} x_n = 0$.

(b) For relevant reading, see Chapter 8 of the subject guide.

(i) As in Definition 8.19 of the subject guide, we say that a set $U \subseteq \mathbb{R}$ is *open* if and only if for any $y \in U$ there exists $\varepsilon = \varepsilon(y) > 0$ such that $(y - \varepsilon,\ y + \varepsilon) \subseteq U$.

(ii) As in Definition 8.20 of the subject guide, we say that a set $C \subseteq \mathbb{R}$ is *closed* if, whenever (x_n) is a convergent sequence with $x_n \in C$ for all n, then the limit of the sequence is also in C.

To prove that $Z(f)$ is closed, we have two approaches: either (1) using the definition directly, or (2) by showing that its complement in $\mathbb{R}$ is open.

Method 1: Using the definition of a closed set, let (x_n) be any convergent sequence in $Z(f)$: we want to show that $x = \lim x_n \in Z$. By the definition of $Z(f)$, $f(x_n) = 0$ for all $n \in \mathbb{N}$. Using a standard result of continuity, (see Theorem 8.14 in the subject guide), we therefore have

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0.$$

Since $f(x) = 0$, we have $x \in Z(f)$, as required.

Method 2: Alternatively, we can prove that $U = \mathbb{R} \setminus Z(f)$ is open. By **Theorem 8.19**, a set $C \subseteq \mathbb{R}$ is closed if and only if its complement $\mathbb{R} \setminus C$ is open. We can then conclude that $Z(f)$ is closed.

Let $y \in U$, then $f(y) \neq 0$. Let $\varepsilon = |f(y)| > 0$. By the continuity of f, there is $\delta > 0$ such that if $|z - y| < \delta$ then $|f(z) - f(y)| < \varepsilon$ and therefore, $f(z) \neq 0$. So, $(y - \delta,\ y + \delta) \subseteq U$.

This shows U is open and hence Z is closed.

(c) For relevant reading, see Chapter 11 of the subject guide.

Using the standard technique, we denote the pay-off matrix of the game by A, the value of the game by λ, and the optimal mixed strategy for Player I by $\boldsymbol{p} = (\alpha,\ 1 - \alpha)^T$, where

$$A^T \boldsymbol{p} = \begin{pmatrix} 4 & 3 \\ 2 & 6 \end{pmatrix}^T \begin{pmatrix} \alpha \\ 1 - \alpha \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}.$$

Eliminating λ, we have

$$4\alpha + 2(1 - \alpha) = \lambda = 3\alpha + 6(1 - \alpha),$$

which yields

$$\lambda = \frac{18}{5} \qquad \text{and} \qquad \boldsymbol{p} = \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}.$$

The optimal strategy for Player II is calculated as

$$\boldsymbol{q} = A^{-1} \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 6 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 18/5 \\ 18/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}.$$